The two fundamental equations for the motion of a system of particles

\[ \sum F = m \ddot{a} \quad \sum M_G = \dot{H}_G \]

provide the foundation for three dimensional analysis, just as they do in the case of plane motion of rigid bodies. The computation of the angular momentum \( H_G \) and its derivative \( \dot{H}_G \), however, are now considerably more involved.
The rectangular components of the angular momentum $H_G$ of a rigid body may be expressed in terms of the components of its angular velocity $\omega$ and of its centroidal moments and products of inertia:

\[
H_x = +I_x \omega_x - I_{xy} \omega_y - I_{xz} \omega_z \\
H_y = -I_{yx} \omega_x + I_y \omega_y - I_{yz} \omega_z \\
H_z = -I_{zx} \omega_x - I_{zy} \omega_y + I_z \omega_z
\]

If principal axes of inertia $Gx'y'z'$ are used, these relations reduce to

\[
H_{x'} = I_x \omega_x, \\
H_{y'} = I_y \omega_y, \\
H_{z'} = I_z \omega_z.
\]
In general, the angular momentum \( \mathbf{H}_G \) and the angular velocity \( \mathbf{\omega} \) do not have the same direction. They will, however, have the same direction if \( \mathbf{\omega} \) is directed along one of the principal axes of inertia of the body.

The system of the momenta of the particles forming a rigid body may be reduced to the vector \( m\mathbf{v} \) attached at \( G \) and the couple \( \mathbf{H}_G \). Once these are determined, the angular momentum \( \mathbf{H}_O \) of the body about any given point \( O \) may be obtained by writing

\[
\mathbf{H}_O = \mathbf{r} \times m\mathbf{v} + \mathbf{H}_G
\]
In the particular case of a rigid body constrained to rotate about a fixed point \( O \), the components of the angular momentum \( \mathbf{H}_O \) of the body about \( O \) may be obtained directly from the components of its angular velocity and from its moments and products of inertia with respect to axes through \( O \).

\[
\begin{align*}
H_x &= +I_x \omega_x - I_{xy} \omega_y - I_{xz} \omega_z \\
H_y &= -I_{yx} \omega_x + I_y \omega_y - I_{yz} \omega_z \\
H_z &= -I_{zx} \omega_x - I_{zy} \omega_y + I_z \omega_z
\end{align*}
\]
The principle of impulse and momentum for a rigid body in three-dimensional motion is expressed by the same fundamental formula used for a rigid body in plane motion.

\[ \text{Syst Momenta}_1 + \text{Syst Ext Imp}_{1\rightarrow 2} = \text{Syst Momenta}_2 \]

The initial and final system momenta should be represented as shown in the figure and computed from

\[
\begin{align*}
H_x &= +\bar{I}_x \omega_x - \bar{I}_{xy} \omega_y - \bar{I}_{xz} \omega_z \\
H_y &= -\bar{I}_{yx} \omega_x + \bar{I}_y \omega_y - \bar{I}_{yz} \omega_z \\
H_z &= -\bar{I}_{zx} \omega_x - \bar{I}_{zy} \omega_y + \bar{I}_z \omega_z
\end{align*}
\]

or

\[
\begin{align*}
H'_x &= \bar{I}_x \omega_x, & H'_y &= \bar{I}_y \omega_y, \\
H'_z &= \bar{I}_z \omega_z
\end{align*}
\]
The kinetic energy of a rigid body in three-dimensional motion may be divided into two parts, one associated with the motion of its mass center $G$, and the other with its motion about $G$. Using principal axes $x', y', z'$, we write

$$\mathbf{T} = \frac{1}{2} m \overline{\mathbf{v}}^2 + \frac{1}{2} \left( I_x \overline{\mathbf{\omega}}_x^2 + I_y \overline{\mathbf{\omega}}_y^2 + I_z \overline{\mathbf{\omega}}_z^2 \right)$$

where $\overline{\mathbf{v}} = \text{velocity of the mass center}$
$\overline{\mathbf{\omega}} = \text{angular velocity}$
$m = \text{mass of rigid body}$
$I_x, I_y, I_z = \text{principal centroidal moments of inertia}$. 
In the case of a rigid body *constrained to rotate about a fixed point* $O$, the kinetic energy may be expressed as

$$
T = \frac{1}{2} \left( I_{x'}\omega_{x'}^2 + I_{y'}\omega_{y'}^2 + I_{z'}\omega_{z'}^2 \right)
$$

where $x'$, $y'$, and $z'$ axes are the principal axes of inertia of the body at $O$. The equations for kinetic energy make it possible to extend to the three-dimensional motion of a rigid body the application of the *principle of work and energy* and of the *principle of conservation of energy*. 
The fundamental equations

\[ \Sigma \mathbf{F} = m \mathbf{a} \quad \Sigma \mathbf{M}_G = \dot{\mathbf{H}}_G \]

can be applied to the motion of a rigid body in three dimensions. We first recall that \( \mathbf{H}_G \) represents the angular momentum of the body relative to a centroidal frame \( GX'Y'Z' \) of fixed orientation and that \( \dot{\mathbf{H}}_G \) represents the rate of change of \( \mathbf{H}_G \) with respect to that frame. As the body rotates, its moments and products of inertia with respect to \( GX'Y'Z' \) change continually. It is therefore more convenient to use a frame \( Gxyz \) rotating with the body to resolve \( \mathbf{\omega} \) into components and to compute the moments and products of inertia which are used to determine \( \mathbf{H}_G \).
\[ \Sigma F = m\ddot{a} \quad \Sigma M_G = \dot{H}_G \]

\( \dot{H}_G \) represents the rate of change of \( H_G \) with respect to the frame \( GX'Y'Z' \) of fixed orientation, therefore

\[ \dot{H}_G = \left( \dot{H}_G \right)_{Gxyz} + \Omega \times H_G \]

where \( H_G \) = angular momentum of the body with respect to the frame \( GX'Y'Z' \) of fixed orientation

\( \left( \dot{H}_G \right)_{Gxyz} \) = rate of change of \( H_G \) with respect to the rotating frame \( G_{xyz} \)

\( \Omega \) = angular velocity of the rotating frame \( G_{xyz} \)
\[ \Sigma F = m \ddot{a} \quad \Sigma M_G = \dot{H}_G \]

\[ \dot{H}_G = \left( \dot{H}_G \right)_{Gxyz} + \Omega \times H_G \]

Substituting \( \dot{H}_G \) above into \( \Sigma M_G \),

\[ \dot{M}_G = \left( \dot{H}_G \right)_{Gxyz} + \Omega \times H_G \]

If the rotating frame is attached to the body, its angular velocity \( \Omega \) is identical to the angular velocity \( \omega \) of the body.

Setting \( \Omega = \omega \), using principal axes, and writing this equation in scalar form, we obtain Euler’s equations of motion.
In the case of a rigid body constrained to rotate about a fixed point \( O \), an alternative method of solution may be used, involving moments of the forces and the rate of change of the angular momentum about point \( O \).

\[
\sum M_O = (\dot{H}_O)_{Oxyz} + \Omega \times H_O
\]

where \( \sum M_O \) = sum of the moments about \( O \) of the forces applied to the rigid body

\( H_O \) = angular momentum of the body with respect to the frame \( OXYZ \)

\( (\dot{H}_O)_{Oxyz} \) = rate of change of \( H_O \) with respect to the rotating frame \( Oxyz \)

\( \Omega \) = angular velocity of the rotating frame \( Oxyz \)
When the motion of gyroscopes and other axisymmetrical bodies are considered, the Eulerian angles $\phi$, $\theta$, and $\psi$ are introduced to define the position of a gyroscope. The time derivatives of these angles represent, respectively, the rates of precession, nutation, and spin of the gyroscope. The angular velocity $\omega$ is expressed in terms of these derivatives as

$$\omega = -\dot{\phi} \sin \theta \mathbf{i} + \dot{\theta} \mathbf{j} + (\dot{\psi} + \dot{\phi} \cos \theta) \mathbf{k}$$
The unit vectors are associated with the frame $Oxyz$ attached to the inner gimbal of the gyroscope (figure to the right) and rotate, therefore, with the angular velocity

$$\mathbf{\Omega} = -\dot{\phi} \sin \theta \mathbf{i} + \dot{\theta} \mathbf{j} + \dot{\phi} \cos \theta \mathbf{k}$$

$$\mathbf{\omega} = -\dot{\phi} \sin \theta \mathbf{i} + \dot{\theta} \mathbf{j} + (\dot{\psi} + \dot{\phi} \cos \theta) \mathbf{k}$$
Denoting by $I$ the moment of inertia of the gyroscope with respect to its spin axis $z$ and by $I'$ its moment of inertia with respect to a transverse axis through $O$, we write

$$\mathbf{H}_O = -I'\dot{\phi}\sin \theta \mathbf{i} + I'\dot{\theta} \mathbf{j} + I(\dot{\psi} + \dot{\phi}\cos \theta) \mathbf{k}$$

Substituting for $\mathbf{H}_O$ and $\Omega = -\dot{\phi}\sin \theta \mathbf{i} + \dot{\theta} \mathbf{j} + \dot{\phi}\cos \theta \mathbf{k}$ into

$$\Sigma \mathbf{M}_O = (\mathbf{H}_O)_{Oxyz} + \Omega \times \mathbf{H}_O$$

leads to the differential equations defining the motion of the gyroscope.
In the particular case of the \emph{steady precession} of a gyroscope, the angle \( \theta \), the rate of precession \( \dot{\phi} \), and the rate of spin \( \dot{\psi} \) remain constant. Such motion is possible only if the moments of the external forces about \( O \) satisfy the relation

\[
\sum M_O = (I \omega_z - I' \dot{\phi} \cos \theta) \dot{\phi} \sin \theta \mathbf{j}
\]

i.e., if the external forces reduce to a couple of moment equal to the right-hand member of the equation above and applied \textit{about an axis perpendicular to the precession axis and to the spin axis}. 