Morlet wavelet

Key formulae: Morlet wavelet
Normalization constant $c_\psi$ is tabulated in the text.

- The scaled wavelet
  \[ \Psi_M(t, \omega) = \frac{\omega}{\sqrt{c_\psi}}(e^{2i\pi\omega t} - e^{-\frac{z_0^2}{2}})e^{-2\pi^2\omega^2t^2/z_0^2} \]

- Transform
  \[ \tilde{u}_M = \int u(t')\Psi_M((t - t'), \omega)dt' \]

- Inverse transform
  \[ u(t) = \int \int \Psi_M((t - t'), \omega)\tilde{u}_M(t', \omega) \frac{d\omega}{\omega} dt \]

- Parseval (Plancherel)
  \[ \int dt \frac{u^2}{2} = \frac{1}{2} \int \int |\tilde{u}_M|^2 \frac{d\omega}{\omega} dt \]

I prefer the logarithmic-scale integration for inverse and Parseval; absorbing $c_\psi$ in the definition of $\Psi_M$ simplifies the formulae; $\omega$ is the frequency of the local oscillation.

The Morlet wavelet is complex-valued, and consists of a Fourier wave modulated by a Gaussian envelope of width $z_0/\pi$:

\[ \psi_{M,z_0}(t) = (e^{2i\pi t} - e^{-\frac{z_0^2}{2}})e^{-2\pi^2t^2/z_0^2} \]  \(1\)

The envelope factor $z_0$ controls the number of oscillations in the wave packet; a value of $z_0 = 5$ is generally adopted, with the result shown elsewhere in these pages. The correction factor $e^{-\frac{z_0^2}{2}}$, making the wavelet admissible, is very small for $z_0 \geq 5$ and often neglected. The Fourier transform is

\[ \hat{\psi}_{M,z_0} = \frac{z_0}{2\sqrt{\pi}}e^{-\frac{z_0^2}{4}(1+\omega^2)}(e^{-z_0^2\omega} - 1) \]  \(2\)
There is apparently no closed-form expression of $c_\psi$, but numerical integration presents no difficulty and yields the values shown in Table 1.

Instead of the usual dilation factor $a$, we adopt its inverse $\omega = 1/a$, which is its own equivalent frequency. For normalization, we adopt

$$\Psi_M(t\omega) = \frac{\omega}{\sqrt{c_\psi}} \psi_{M,z_0}(t, \omega).$$

Then, the three basic formulae are, for the transform

$$\bar{u}_M = \int u(t') \Psi_M^*(\omega(t - t'), \omega) dt',$$

for the energy of the signal

$$\int dt \frac{u^2}{2} = \frac{1}{2} \int \int |\bar{u}_M|^2 \frac{d\omega}{\omega} dt,$$
Figure 2: Norm (top) and phase (bottom) of the Morlet transform of our signal. Also shown is the cone of influence of the end-points.

and for the inverse transform

\[ u(t) = \int \int \Psi_M((t - t'), \omega) \bar{u}_M(t', \omega) \frac{d\omega}{\omega} dt. \]  

\[ \text{(6)} \]

Fig. 2 shows the norm and phase of the Morlet transform of our signal.