Mexican hat wavelet

### Key formulae: Mexican hat wavelet

- **Gaussian filter**
  \[ F_\sigma(t) = \frac{1}{2\sqrt{\pi\sigma}} e^{-\frac{t^2}{4\sigma}} \]

- **The scaled wavelet**
  \[ \psi_2(\sigma, t) = \frac{dF_\sigma}{d\sigma} = \frac{d^2F_\sigma}{dt^2} \]

- **Transform**
  \[ \tilde{u}_2(\sigma, t) = \int u(t')\psi_2(\sigma, t - t') dt' \]

- **Compatibility condition**
  \[ \partial_\sigma \tilde{u}_2 = \partial_t^2 \tilde{u}_2 \]

- **Inverse transform**
  \[ u(t) = 4 \int \int \tilde{u}_2(\sigma, t') \psi_2(\sigma, t - t') d\sigma dt' \]

- **Multiresolution**
  \[ \tilde{U} = -\sigma \tilde{u}_2 \]

- **Local reconstruction**
  \[ u(t) = \int_0^\infty \tilde{U} \frac{d\sigma}{\sigma} \]

- **Parseval (Plancherel)**
  \[ \int \frac{1}{2} u^2 \, dx = \int \frac{ds}{s} \int 2 \tilde{U}^2 \, dx \]

The normalization constant \( c_{p,si} \) is absorbed in the definition of \( \psi_2 \); the scale \( \sigma \) corresponds to a frequency \( \omega = 1/2\pi\sqrt{\sigma} \); logarithmic scale integration for inverse and Parseval are natural; and the compatibility equation facilitates analytical manipulations.

Starting from the normalized Gaussian filter

\[ F_\sigma(t) = \frac{1}{2\sqrt{\pi\sigma}} e^{-\frac{t^2}{4\sigma}} \]  \( \text{(1)} \)
in which $\sigma = a^2/2$, its derivative with respect to $\sigma$ (second time derivative) gives the Mexican hat wavelet:

$$\psi_2(\sigma, t) = \frac{dF_\sigma}{d\sigma} = \left(\frac{t^2}{4\sigma^2} - \frac{2}{\sigma}\right)F_\sigma(t) = \frac{d^2F_\sigma}{dt^2}. \quad (2)$$

It is a difference of Gaussian filters of different scales (band-pass filtering), divided by the scale difference. Rather than energy normalization, this variant favors the relation to Gaussian filtering and the simple formulae below, including the compatibility equation.

The Fourier transform of $\psi_2$ is

$$\hat{\psi}_2(\sigma, \omega) = -4\pi^2\omega^2 e^{-4\pi^2\omega^2\sigma}. \quad (3)$$

For a signal $u$, the Mexican hat wavelet transform is written as

$$\tilde{u}_2(\sigma, t) = \int u(t')\psi_2(t - t', \sigma) dt' = \frac{d^2}{dt^2}(F_\sigma * u) \quad (4)$$

Then the energy of the signal is given by

$$\int \frac{|u|^2}{2} dt = \int \int 2 |\sigma\tilde{u}_2|^2 \frac{d\sigma}{\sigma} dt \quad (5)$$

The inverse transform reconstructs the signal from its wavelet coefficients by the relation

$$u(t) = 4 \int \int \sigma\tilde{u}_2(\sigma, t') \psi_2(t - t', \sigma) d\sigma dt'. \quad (6)$$

For the Mexican hat, a simpler exact alternative exists in the form

$$u(t) = -\int_0^\infty \tilde{u}_2(\sigma, t) d\sigma = \int_0^\infty \tilde{U}(\sigma, t) \frac{d\sigma}{\sigma} \quad (7)$$

when we introduce the multiscale decomposition of $u$ as

$$\tilde{U}(\sigma, t) = -\sigma\tilde{u}_2(\sigma, t) \quad (8)$$

Because $\tilde{U}$ can be interpreted as the ‘$u$’ at scale $\sigma$ and point $x$, and because of its relation to energy (see below), I prefer it for plotting purposes. A comparison with the orthodox presentation appears in the web pages.

For plotting purposes, the conversion from $\sigma$ to an equivalent frequency is needed. Taking the wavelet transform of $\cos(2\pi\omega t)$, two simple conversions can be adopted: the peak of the compensated energy spectrum corresponds to $\omega\sqrt{\sigma} = 1/\pi\sqrt{8}$, or the centroid of the spectrum coincides with $\omega\sqrt{\sigma} = 1/2\pi$. The second alternative is adopted here, with the largest $\omega$ equal to the Nyquist frequency of our discrete signal.
Figure 1: Mexican hat wavelet map $\tilde{U}$ (bottom) and signal (top).

The example signal and its Mexican hat wavelet map are shown on Fig. 1. The largest wavelet coefficients (red for maxima, blue for minima) identify individual extrema of the signal. The near-periodicity of one contribution to the signal is visible in the energetic contributions along a horizontal strip of the map. The noise is intermittently distributed in the top third of the map, corresponding to larger frequencies. Additional features are discussed in connection with the Morlet transform.

Parseval relation, spectra

The Parseval/Plancherel formula takes the form

$$\int \frac{1}{2} \tilde{u}^2 \, dx = \int \int 2s \tilde{u}^2 \, ds \, dx = \int \int 2 \tilde{U}^2 \frac{ds}{s} \, dx. \quad (9)$$

Again, we see that $\tilde{U}$ (squared) is proportional to the energy density (per unit time and per logarithmic frequency) of the signal.

Cone of influence and end effects

For a wavelet located at a fixed time, increasing its scale from some very small value gradually brings a larger part of the signal into the wavelet’s view, generating the cone of influence of the wavelet. Signal values within the cone of influence affect the wavelet coefficient at that scale. If the wavelet is located near either end of a signal, it will see whatever information is located beyond the end-points, or the lack of information: the corresponding wavelet
coefficients are of questionable value. Various techniques such as wrap-around (implied periodicity of the signal), mirror symmetry or zero-padding do not entirely eliminate this problem, and results should be interpreted accordingly. Periodicity is used throughout the examples below. The cone of influence is the envelope, for some suitably low threshold (2% of maximum in this case), of the wavelet coefficients $\tilde{u}$ of a Dirac function located at an end point. It is recommended that the cone of influence of end-points be shown as a reminder that the wavelet coefficients within the cone should be interpreted with caution.