FIELD RECONSTRUCTION
FROM SINGLE SCALE CONTINUOUS WAVELET COEFFICIENTS

JACQUES LEWALLE

Department of Mechanical and Aerospace Engineering
Syracuse University, Syracuse NY 13244, USA.
jlwalle@syr.edu

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The redundancy of continuous wavelet transforms implies that the wavelet coefficients are not independent of each other. This interdependence allows the reconstruction or approximation of the wavelet transform, and of the original field, from a subset of the wavelet coefficients. Contrasting with lines of modulus maxima, known to provide useful partition functions and some data compaction, the reconstruction from single-scale coefficients is derived for the Hermitian family of wavelets. The formula is exact in the continuum for \(d\)-dimensional fields, and its limitations under discretization are illustrated.

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1. Introduction

At the outset, we distinguish the application of continuous wavelet transforms (CWTs) to continuous fields and their governing partial-differential equations, \(^{14,13}\), which is our goal, from discretized solutions used here for illustration purposes only. The perspective adopted here is not computational, and issues of algorithm performance are accordingly marginalized.

In sharp contrast to the economy of representation provided by orthogonal wavelets, CWTs are known to be vastly redundant, \(^{3,26}\). This implies that the wavelet coefficients are not independent of each other. The corresponding relation has traditionally been written in terms of the reproducing kernel, \(^{3,26,12}\), with a linear partial-differential compatibility equation, \(^{19,14}\) as an attractive alternative in some cases, as seen below.

The redundancy invites attempts to reconstruct or approximate the data from subsets of the wavelet coefficients. Some data compaction with continuous wavelets has been achieved with the Lines of Modulus Maxima (LMMs), \(^{25,20,21,22}\). The idea starts with the observation that the transition between extrema of the coefficients at any given scale is very smooth, and can be interpolated with good accuracy:
most of the information about the signal is therefore captured by the location and
values of the coefficients along the LMMs. As a data compaction tool, this
approach has been superseded by orthogonal wavelet thresholding. Thus unpromising
for data storage and transmission or for computer simulation, continuous wavelets
nonetheless present advantages for frequency resolution in the analysis of data, 6,
for the definition of partition functions and other applications based on lines of
modulus maxima, 1, for the connection to structure functions, 12, and for formal

This paper presents a new tool relevant to the latter. The application of the
wavelet transform (continuous or orthogonal) to nonlinear dynamical equations
gives complicated expressions coupling the coefficients across scales, 7,14. Such mul-
tiscale dynamics require considerable computational resources and creativity, and
have so far been an obstacle to analytical developments. The main result derived
below is the reconstruction of the field from single-scale coefficients (exact recon-
struction in the continuum, partial reconstruction in the case of the discrete signals
used as examples). This approach is developed for the Hermitian family of wavelets,
i.e. derivatives of the Gaussian filter, of which the Mexican hat is the first and best-
known member. The derivation takes advantage of the very simple compatibility
equation obeyed by the wavelet coefficients in location-scale (or time-frequency)
space.

Section 2 establishes notations and collects some published (Mexican hat
wavelet) and new relations (higher-order Hermitian wavelets). In particular, our
results depend on the multiscale representation of fields, a convolution-free version
of the inverse continuous wavelet transform, 19,17,16,14,13. Several options are intro-
duced in Section 3 to reconstruct the entire field from the wavelet coefficients at a
single scale. This reconstruction is exact in the continuum, accurate towards larger
scales or if the complex coefficients saved from the original wavelet transform are
used, and otherwise limited by numerical noise toward small scales.

2. The Hermitian wavelet transform.
2.1. Mexican hat wavelet.

Notations, terminology and some basic operations occupy us first. The d-
dimensional Mexican-hat wavelet derives from the Gaussian filter

\[ F_s(x) = \frac{1}{(2\sqrt{\pi s})^d} e^{-x^2/4s}. \]  

(2.1)

The \( L^2 \) wavelet normalization favored in the literature, 26, is not the only choice,
25,8,27. Here, we will use two \( L^1 \) wavelet normalizations, anticipating that some
operations and equations, such as diffusion and iterative filtering, are easier to
manipulate with one, while others, such as scale dominance and the discretization
of the scale axis, benefit from the alternative.
Single-scale wavelet reconstruction

The first normalization corresponds to the wavelet

\[ \psi_s = \partial_s F_s = \nabla^2 F_s, \tag{2.2} \]

i.e. the difference of two Gaussian filters of nearby scales divided by the scale difference \( ds \). It is admissible, \( 3, 26 \), since \( \int \psi_s \, dx = 0 \). The wavelet transform of a field \( u \) is obtained by convolution

\[ \tilde{u}(x, s, t) = \tilde{\psi}_s * u = \partial_s u^2 = F_s * \nabla^2 u. \tag{2.3} \]

For example, in the case of \( u_{\text{cos}} = \cos(x/\sqrt{S}) \), we have

\[ \tilde{u}_{\text{cos}} = -\frac{1}{S} e^{-s/S} \cos(x/\sqrt{S}), \tag{2.4} \]

and for \( u_{\text{bell}} = \frac{1}{2 \sqrt{\pi S}} e^{-x^2/4S} \)

\[ \tilde{u}_{\text{bell}} = e^{-x^2/4(S+s)} \sqrt{\frac{S}{S+s}} \left( \frac{x^2 - 2(s + S)}{4(s + S)^2} \right). \tag{2.5} \]

The admissibility condition implies that the wavelet coefficients at any scale have zero mean, and that the transform of a constant is zero. For the Mexican hat, Eq.\( (2.2) \) shows further that the transform of any field with vanishing Laplacian is zero. The relation between \( s \)- and \( x \)-derivatives makes the Mexican hat wavelet an elementary solution of the diffusion equation. Since the diffusion equation applies also to the wavelet transform, it is also the compatibility equation, \( 17, 14 \), for a field \( \tilde{u}(x, s) \) (with \( 0 < s < \infty \)) to derive from \( \tilde{u}(x, 0) = \nabla^2 u \) by Gaussian filtering:

\[ \tilde{u}(x, s) = F_s * \tilde{u}(x, 0) = F_s * \nabla^2 u \quad \leftrightarrow \quad \partial_s \tilde{u}_s = \nabla^2 \tilde{u}_s. \tag{2.6} \]

As a consequence of this relation, the wavelet transform at any scale can be regarded as ‘initial’ condition for the transform at all larger scales. In particular, the entire wavelet transform can be reconstructed from \( \tilde{u}(x, 0) \) by Gaussian filtering.

The alternative normalization is based on the logarithmic scale difference \( -ds/s \), and the wavelet becomes

\[ w_s = -s\psi_s. \tag{2.7} \]

The corresponding transform

\[ \tilde{U}_s = w_s * u = -s \tilde{u}_s \tag{2.8} \]

is called the multiscale representation, \( 4 \), as explained in Section 2.3. Note that instead of Eq.\( (2.6) \), \( \tilde{U} \) obeys the compatibility equation

\[ \partial_s \tilde{U}_s = \frac{\tilde{U}_s}{s} - \nabla^2 \tilde{U}_s. \tag{2.9} \]

With these notations, the Parseval relation becomes

\[ \int_{-\infty}^{\infty} \frac{1}{2} u^2 \, dx = \int_{0}^{\infty} ds \int_{-\infty}^{\infty} 2s^2 \tilde{u}_s^2 \, dx = \int_{-\infty}^{\infty} \frac{ds}{s} \int_{-\infty}^{\infty} 2 \tilde{U}_s^2 \, dx. \tag{2.10} \]
It can be interpreted as distributing the total energy among the wavelet coefficients; it can be shown that \( \int 2 \hat{U}_s^2 \, dx \) is the wavelet (smoothed) version of the compensated Fourier spectrum \( \kappa \hat{E}(\kappa) \).

For \( u_{\cos} \), averaging over one period, we have a mean wavelet spectrum per unit length

\[
\bar{\varepsilon}_{\cos} = \frac{1}{2\pi \sqrt{S}} \int_0^{2\pi \sqrt{S}} 2s \bar{u}_{\cos}^2 \, dx = \frac{s}{S^2} e^{-2s/S} \tag{2.11}
\]

and

\[
\int_0^\infty \bar{\varepsilon}_{\cos} \, ds = \frac{1}{4} = \frac{1}{2\pi \sqrt{S}} \int_0^{2\pi \sqrt{S}} \frac{u_{\cos}^2}{2} \, dx, \tag{2.12}
\]

which verifies the normalization. For \( u_{\text{bell}} \), the mean wavelet spectrum over the entire real line is

\[
\bar{\varepsilon}_{\text{bell}} = \int_{-\infty}^{\infty} 2s \bar{u}_{\text{bell}}^2 \, dx = \frac{3sS\sqrt{2\pi}}{8(s + S)^{5/2}} \tag{2.13}
\]

with another check on normalization

\[
\int_0^\infty \bar{\varepsilon}_{\text{bell}} \, ds = \int_{-\infty}^{\infty} \frac{u_{\text{bell}}^2}{2} \, dx = \frac{\pi S}{2}. \tag{2.14}
\]

For purposes of illustration, a short velocity signal \( u \) and its \( \hat{U} \) transform are shown on Fig. 1. The non-stationary streamwise velocity is associated with a turbulent spot in a flat-plate boundary layer undergoing bypass transition. The ‘frozen turbulence’ conversion from time to space, and related units, are inconsequential for the limited scope of this illustration; arbitrary but consistent units are used for \( \kappa \) and \( x \). \( \hat{U} \) varies smoothly in both space \( (x) \) and wavenumber \( (\kappa) \). Regions of large positive (red) and negative (blue) values are distributed intermittently in the \( (x - \kappa) \) half-plane. The markers (+ on the figure) will be used in relation with further processing below.

### 2.2. Higher-order Hermitian wavelets.

Less widely used are the higher-order derivatives of Gaussian: the Hermitian wavelets. The two normalizations are extended to the entire family. The wavelets of order \( n \) are defined as

\[
\psi_{s,n} = \partial_x^n F_s = \partial_x^n \psi_{s,n-1} \tag{2.15}
\]

and

\[
w_{s,n} = \frac{(-s)^n}{\Gamma(n)} \psi_{s,n}. \tag{2.16}
\]

The Mexican hat corresponds to \( n = 1 \). The wavelet transforms are defined, respectively, as

\[
\tilde{u}_{s,n} = \psi_{s,n} * u = \partial_x \tilde{u}_{s,n-1} \tag{2.17}
\]
Single-scale wavelet reconstruction

Fig. 1. A signal $u$ of a turbulent spot, and its multiscale decomposition $\tilde{U}_s$ (red for positive values, blue for negative).

and

$$\tilde{U}_{s,n} = w_{s,n} * u = \frac{(-s)^n}{\Gamma(n)} \tilde{u}_{s,n}. \quad (2.18)$$

It is easy to show that the frequency resolution of the wavelet improves with $n$. The multiscale coefficients $\tilde{U}_{s,4}$ are shown on Fig. 2.

It can be shown easily (e.g. for $u_{\cos}$, and by extension for any Fourier-transformable field) that the Parseval relation takes the form

$$\frac{1}{2} \int u^2 dx = \int dx \int ds \frac{(2s)^{2n}}{s \Gamma(2n)} \tilde{u}_{s,n}^2 = \frac{\sqrt{n} \Gamma(n)}{\Gamma(n + 1/2)} \int dx \int ds \frac{\tilde{u}_{s,n}^2}{s}. \quad (2.19)$$

2.3. Inverse transform.

Field reconstruction is given by the inverse wavelet transform, 3,26. With our notations

$$u = 4 \int_0^\infty s \tilde{u}_s * \tilde{u}_s \, ds, \quad (2.20)$$

which requires a convolution as well as integration over all scales. It is often overlooked that, strictly speaking, this reconstruction is exact within the addition of any field of vanishing Laplacian, for which the wavelet coefficients are identically zero. This situation is similar to the calculation of vorticity-induced velocity from the Biot-Savart relation in fluid dynamics, where the addition of a potential field is required to meet given boundary conditions.
For the Mexican hat wavelet, a convolution-free (local) alternative to the inverse transform exists, \(^{19,16,13}\) in the form
\[
\int_0^\infty \tilde{u}_s \, ds = \int_0^\infty \partial_s u^s \, ds = \left[u^s\right]_0^\infty = -u \tag{2.21}
\]
for a field \(u\) of vanishing mean in any dimension \(d\). This is easily verified for \(u_{\cos}\), and by extension for any Fourier decomposition of a signal. It was shown, \(^{14}\), that Eq. (2.21) is equivalent to the Biot-Savart formula and is one representation of the inverse Laplacian in \(R^d\) (i.e. the integral of the Gaussian filter over all scales gives Green’s function for the Poisson equation in 2- and 3-\(d\)).

Alternatively, at any location \(x\) and for any component of \(u\),
\[
u = \int \tilde{U}_s \, \frac{ds}{s}. \tag{2.22}
\]
If \(u\) is a velocity component, so is also \(\tilde{U}_s\). Thus \(\tilde{U}_s\) is one possible version of the local velocity at length scale \(\ell = 2\pi \sqrt{s}\). Following Eyink, \(^4\), we will call it the ‘multiscale’ decomposition of the field. Not only is \(2U^2\) the local spectral (logarithmic) energy density (Eq.(2.10)), but \(\tilde{U}_s\) is also the basic building block to reconstruct \(u\) from its wavelet coefficients at each point \(x\) (Eq.(2.22)). The logarithmic scale, natural for orthogonal wavelets, \(^{26}\), for shell models, \(^2\) and for multifractal fields, \(^{28}\), is built into the \(\tilde{U}_s\) representation.

Similarly, for the higher-order Hermitian wavelets, we have
\[
u = \int \frac{ds}{s} (2s)^{2n} \psi_{s,n} \ast \tilde{u}_{s,n} = \int \frac{ds}{s} \frac{(-s)^n}{\Gamma(n)} \tilde{u}_{s,n}, \tag{2.23}
\]
and
\[ u = \int \frac{ds}{s} \tilde{U}_{s,n} = \frac{2 \sqrt{n} \Gamma(n)}{\Gamma(n + 1/2)} \int \frac{ds}{s} \omega_{s,n} * \tilde{U}_{s,n}. \] (2.24)

Note the absence of \( s \)-dependent weighting in the logarithmic integrals when the multiscale form \( \tilde{U}_{s,n} \) is used.

3. Field reconstruction from a single scale

Let us take \( s \) as an arbitrary fixed scale of \( \tilde{u}_s \), from which the field will be approximated, and focus on Eq.(2.21). Then, we split the \( s \)-integral into the two ranges \( 0 < s' < s \) and \( s < s' < \infty \). It is obvious that the wavelet coefficients at larger scale \( s' > s \), and therefore the filtered coarse-grained field \( u^{\geq s} = - \int_s^\infty \tilde{u}_{s'} ds' \) can be constructed by diffusion to larger \( s \) (Eq.2.6), i.e. by the spatial convolution
\[ \tilde{u}_{s'} = F_{s'} * \tilde{u}_s. \] (3.1)

Then,
\[ u^{\geq s} = - \left( \int_0^\infty F_z dz \right) * \tilde{u}_s = BS * \tilde{u}_s = - \frac{1}{s} BS * \tilde{U}_s, \] (3.2)
where we recognize the the Biot-Savart kernel \( BS = - \frac{1}{4\pi} \frac{1}{\|x\|} \) for 3-d fields and \( BS = \frac{1}{2\pi} \ln \|x\| \) for 2-d fields. The Fourier-space formulation used below is also an accurate option in this range of scales.

Less obvious is the reconstruction of smaller scales \( s' < s \) from \( \tilde{u}_s \). The root of the difficulty is that the operation amounts to an inverse diffusion problem, known for non-unique solutions and numerical instability. From a numerical perspective, loss of accuracy can be expected for large scale ratios, and/or if round-off introduces noise in the calculations. The results will be presented for two procedures, one (noise-free) in which the complex Fourier coefficients \( \tilde{u}_s \) are saved during the initial wavelet transform and the other in which only the real part of \( \tilde{u}_s \) is used.

Three methods are documented: analytic continuation in Fourier space, where inverse diffusion becomes a multiplication by \( e^{\kappa^2(s-s')} \); analytic continuation in the wavelet half-plane by Taylor series; and a corresponding series expansion using the higher-order Hermitian wavelets at scale \( s \). The formulae in the continuum are exact, and the limitations displayed by the illustrations are interpreted as purely numerical. The results for the three methods (explained next) are shown on Fig. 3. In all cases, the dashed line shows the common scale from which reconstruction is attempted; all wavelet coefficients are recalculated from this single scale.

Analytic continuation can be carried out in principle in the Fourier domain, where
\[ \tilde{u}_s = \int e^{-2i\pi \kappa x} \tilde{u}_s dx = -e^{-\kappa^2 s} 4\pi^2 \kappa^2 \int e^{-2i\pi \kappa x} u dx. \] (3.3)

Then, for \( s' < s \) we have
\[ \tilde{u}_{s'} = e^{\kappa^2(s-s')} \tilde{u}_s. \] (3.4)
The Fourier coefficients at any filtered scale have not disappeared altogether, but have been damped exponentially. Numerical implementation on a discretized field is very accurate as long as no round-off noise is introduced in the procedure starting with $\hat{u}_s$ (Fig. 3, top left). The small errors introduced by taking the inverse Fourier
transform and then its real part to produce \( u_s \), are amplified exponentially. Only a few scales, of the order of \( s/2 \), were reasonably accurate, the others are not shown on Fig. 3 (top left).

A local alternative is based on Taylor series of the wavelet coefficients, with \( s' = s - z \),

\[
\tilde{u}_{s'} = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!} [\partial^n \tilde{u}_s]_s = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!} [\nabla^{2n} \tilde{u}_s]_s
\]

which is integrable to \( s_0 \). Note that the term in square brackets is independent of \( s_0 \). This reconstruction of small scale features is again based on their remaining traces in \( \tilde{u}_s \) in each \( x \)-vicinity.

Indeed, for \( \tilde{u}_{\cos} \), we get

\[
\tilde{u}_{\cos}(s', x) = -\frac{1}{S} e^{-s'/S} \sum_{n=0}^{\infty} \frac{(s' - s)^n}{n!} \frac{(-1)^n}{S^n} \cos \left( \frac{x}{\sqrt{S}} \right) = -\frac{1}{S} e^{-s'/S} e^{-(s'-s)/S} \cos \left( \frac{x}{\sqrt{S}} \right).
\]

Superposition of such expansions for arbitrary Fourier modes extends exactly to any Fourier-transformable function.

We conclude that the wavelet coefficients at a single scale \( s \) and in the continuum of \( x \)-space, contain the information required for reconstruction of coefficients at all scales:

\[
u = \sum_{n=0}^{\infty} \frac{(-s)^{n+1}}{\Gamma(n)} \partial^n \tilde{u}_s - BS \ast \tilde{u}_s = \sum_{n=0}^{\infty} \frac{(-s)^{n+1}}{\Gamma(n)} \nabla^{2n} \tilde{u}_s - BS \ast \tilde{u}_s
\]

Again, two approaches were adopted, taking the successive Laplacians in Fourier space on \( \tilde{u}_s \), or using high-order spatial finite differencing of \( \tilde{u}_s \). The results as shown include 29 terms each, beyond which numerical noise becomes dominant. Reconstruction appears to be accurate down to \( s = 20 \).

The numerical factor in this form of the small scale contribution \((0 < s' < s)\) to \( u \) indicates a relationship with the higher-order Hermitians. Simple algebra gives the alternative form

\[
u = \sum_{n=1}^{\infty} \frac{U_{s,n}}{n} - \frac{1}{s} BS \ast \tilde{U}_s.
\]

Compared to the Taylor series, a slightly less extended range is obtained if \( \tilde{U}_{s,n} \) is expanded from \( \tilde{u}_s \) (Fig. 3, bottom right), with 18 terms included in the series before significant numerical noise affects the result. However, if \( \tilde{U}_{s,n} \) is calculated from the original \( u \), no such limitations arise and near-perfect reconstruction is obtained from the first 80 terms in the series (Fig. 3, bottom left). The simple addition of \( \tilde{U}_{s,n} \)-terms reconstructs the small scales from \( s \). In fact, the equivalence of scale \( s \) and wavenumber can be defined from the spectrum of \( u_{\cos} \); the dominant scales (peak of \( \tilde{U}_{s,n} \)) and scale of the centroid of the energy spectrum verify the relation

\[
\frac{1}{\kappa} = 2\pi \sqrt{s/n}.
\]
Therefore, the frequency content increases as $\sqrt{n}$; with 80 terms included in the reconstruction above, one would expect reconstructed scales over a range of the order of 9:1, as observed.

### 3.1. Discussion

The imperfect spectral resolution of wavelets leaves traces of all frequencies in the single-scale coefficients. The three methods presented above for the recovery of this information, apply specifically to the Hermitian family of wavelets. Indeed all three methods depend on the wavelet coefficients satisfying the compatibility equation, by which large scale coefficients are derived from smaller-scale ones by diffusion with $s$ as the time-like variable and unit diffusivity. Thus, from a given scale $s$, the reconstruction of coefficients at larger scales is a simple problem; the reconstruction of coefficients at smaller scales is an inverse diffusion problem.

The Fourier-representation, with its exponential amplification (inverse diffusion) towards small scales, is well known. As an analytical tool, it negates the advantages of the local-spectral decomposition of wavelets. The Taylor series in the $s$-direction is conceptually simple and strictly local. Variants of the Taylor series follow from the equivalence of $s$-derivatives and spatial Laplacians (diffusion), and from the introduction of higher-order Hermitian wavelets. The formulae are based on series expansions in the $s$-direction, and are exact for any dimensionality $d$. The 1-dimensional examples illustrate some of the shortcomings of discretization and numerical round-off, but also confirm the value of the formulae. Although it is unclear whether any of these single-scale expansions is simpler than the multiscale interactions implied by nonlinear governing equations, it is likely that asymptotic estimates in the vicinity of singularities might benefit from this approach.

### References