4.7 Lagrange’s Equations

Defining work, energy and virtual displacements and work we will learn an alternate method of deriving equations of motion

Generalized coordinates: 2 not 4!

Recall equations (1.63) and (1.64)
Definitions (from Dynamics)

Kinetic Energy: \[ T = \frac{1}{2} m \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} = \frac{1}{2} m \dot{\mathbf{r}}^T \dot{\mathbf{r}} \]

Work Done by a force: \[ W_{1\rightarrow 2} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot d\mathbf{r} \]

\( \mathbf{r}_0 \) a reference position then the potential energy is \[ V(r) = \int_{\mathbf{r}_1}^{\mathbf{r}_0} \mathbf{F} \cdot d\mathbf{r} \]
Strain Energy in a Spring

Strain energy (elastic potential energy) for a spring: \( F = -kx \)

\[
V(x) = \int_{x}^{0} F(\eta) \, d\eta = \int_{x}^{0} -k\eta \, d\eta = \frac{1}{2} kx^2
\]

which is the area under the \( F(x) \) vs \( x \) curve
Strain energy in an elastic material

The variation of $dx$, denoted $\delta(dx)$, is given by

$$\delta(dx) = \frac{\partial u(x,t)}{\partial x} dx = \varepsilon(x,t) dx$$

The axial stress is $\sigma(x,t) = \frac{P(x,t)}{A(x)} = E\varepsilon(x,t)$

so $P = EA\varepsilon$
Strain energy continued

\[ dV = \frac{1}{2} P(x,t)\delta(dx) = \frac{1}{2} P(x,t)\varepsilon(x,t)dx \]

\[ = \frac{1}{2} \left[ EA(x)\varepsilon(x,t) \right] \varepsilon(x,t)dx \]

\[ = \frac{1}{2} EA(x)\varepsilon^2(x,t)dx \]

Integrating yields the strain energy for axial tension in a bar element:

\[ V = \frac{1}{2} E \int_{0}^{\ell} A(x)\varepsilon^2(x,t)dx \]

\[ = \frac{1}{2} E \int_{0}^{\ell} A(x) \left( \frac{\partial u(x,t)}{\partial x} \right)^2 dx \]
Virtual Reality  (actually: virtual displacement)

A virtual displacement

Based on variational math

Small or infinitesimal changes compatible with constraints

No time associated with change
Consequence of satisfying the constraint:

Constraint: \( f(\mathbf{r}) = c \), a constant

\[ \Rightarrow f(\mathbf{r} + \delta\mathbf{r}) = c \]

Taylor expansion:

\[ f(\mathbf{r}) + \sum_{i=1}^{n} \left( \frac{\partial f}{\partial x_i} \delta x_i \right) = c \]

\[ \Rightarrow \frac{\partial f}{\partial \mathbf{r}} \cdot \delta\mathbf{r} = 0 \]
Virtual work

Suppose the $i^{th}$ mass is acted on by $\mathbf{f}_i$ with system in static equilibrium

\[ \Rightarrow \delta W_i = \mathbf{f}_i \cdot \delta \mathbf{r}_i = 0, \Rightarrow \]

the principle of virtual work:

\[ \sum_{i=1}^{n} F_i \cdot \delta \mathbf{r}_i = 0 \]

which states that if a system is in equilibrium, the work done by externally applied forces through a virtual displacement is zero:

\[ \Rightarrow \delta V = 0 \]

\[ \Rightarrow V \text{ has an critical value} \]
Dynamic Equilibrium

D'Alembert's Principle $\Rightarrow$ move inertia force to left side and treat dynamics as statics. From Newton's law in terms of change in momentum:

$$\sum F_i = \dot{p} \Rightarrow \left( \sum F_i - \dot{p} \right) = 0$$

This allows us to use virtual work in the dynamic case:

$$\Rightarrow \left( \sum F_i - \dot{p} \right) \cdot \delta r = 0$$

$$(\sum F_i - m\ddot{r}) \cdot \delta r = 0$$
Hamilton’s Principle

\[
\frac{d}{dt}(\ddot{r} \cdot \delta \mathbf{r}) = \dddot{r} \cdot \delta \mathbf{r} + \dot{r} \delta \dot{r}
\]

\[
= \dddot{r} \cdot \delta \mathbf{r} + \delta \left( \frac{1}{2} \dot{r} \cdot \dot{r} \right)
\]

\[
\Rightarrow \sum \dddot{r} \cdot \delta \mathbf{r} = \sum \frac{d}{dt} (\ddot{r} \cdot \delta \mathbf{r}) - \sum \delta \left( \frac{1}{2} \dot{r} \cdot \dot{r} \right), \text{ multiply by } m
\]

\[
\Rightarrow \delta W = \sum m \frac{d}{dt} (\ddot{r} \cdot \delta \mathbf{r}) - \delta T
\]

\[
\Rightarrow \delta T + \delta W = \sum m \frac{d}{dt} (\ddot{r} \cdot \delta \mathbf{r})
\]
Integrate this last expression

\[ \int_{t_1}^{t_2} (\delta T + \delta W) \, dt = \int_{t_1}^{t_2} \sum m \frac{d}{dt}(\dot{\mathbf{r}} \cdot \delta \mathbf{r}) \, dt \]

\[ \int_{t_1}^{t_2} (\delta T + \delta W) \, dt = \sum m \dot{\mathbf{r}} \cdot \delta \mathbf{r}\bigg|_{t_1}^{t_2} = 0 \Rightarrow \text{path indepence} \]

\[ \int_{t_1}^{t_2} (\delta T + \delta W) \, dt = 0, \text{ for conservative forces } \delta W = -\delta V \]

\[ \Rightarrow \delta \int_{t_1}^{t_2} (T - V) \, dt = 0, \text{ Hamilton's principle} \]
Lagrange’s Equation

Let \( \mathbf{r} = \mathbf{r}(q_1, q_2, q_3 \ldots q_n, t) \), \( q_i \) called generalized coordinates

Let \( Q_i = \frac{\partial W}{\partial q_i} \) a generalized force (or moment) \hspace{1cm} (4.143)

The Lagrange formulation, derived from Hamilton's principle for determining the equations of motion are

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i \hspace{1cm} (4.144)
\]
The Lagrangian, $L$

Let $L = (T - U)$, called the Lagrangian

Then (4.145) becomes:

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0, \quad i = 1, 2, \ldots n \quad (1.146)$$

For the (common) case that the potential energy does not depend on the velocity: $\frac{\partial U}{\partial \dot{q}_i} = 0$
Advantages

- Equations contain only scalar quantities
- One equation for each degree of freedom
- Independent of the choice of coordinate system since the energy does not depend on coordinates
- See examples in Section 4.7 pages 332-335
- Useful in situations where $\mathbf{F} = ma$ is not obvious
Example of Generalize Coordinates

How many dof?
What are they?
Are there constraints?

\[
x_1^2 + y_1^2 = \ell_1^2 \quad (x_2 - x_1)^2 + (y_2 - y_1)^2 = \ell_2^2
\]

There are only 2 DOF and one choice is:

\[q_1 = \theta_1 \quad \text{and} \quad q_2 = \theta_2\]
Here $G$ is mass center and $e$ is the distance to the elastic axis. Let $m$ denote the mass of the wing and $J$ the rotational inertia about $G$.

Take the generalized coordinates to be:

\[ q_1 = x(t), \quad q_2 = \theta(t) \]

Called the pitch and plunge model
Computing the Energies

The Kinetic Energy is

$$T = \frac{1}{2} m\dot{x}_G^2 + \frac{1}{2} J\dot{\theta}^2$$

The relationship between $x_G$ and $x$ is

$$x_G(t) = x(t) - e\sin \theta(t)$$

$$\Rightarrow \dot{x}_G(t) = \dot{x}(t) - e\cos \theta(t) \frac{d\theta}{dt} = \dot{x}(t) - e\dot{\theta}\cos \theta(t)$$

So the kinetic energy is

$$T = \frac{1}{2} m[\dot{x} - e\dot{\theta}\cos \theta]^2 + \frac{1}{2} J\dot{\theta}^2$$
Potential Energy and the Lagrangian

The potential energy is:

\[ U = \frac{1}{2} k_1 x^2 + \frac{1}{2} k_2 \theta^2 \]

The Lagrangian is:

\[ L = T - U = \frac{1}{2} m \left( \dot{x} - e\dot{\theta} \cos \theta \right)^2 + \frac{1}{2} J \dot{\theta}^2 - \frac{1}{2} k_1 x^2 - \frac{1}{2} k_2 \theta^2 \]
Computing Derivatives for Eq 1

\[ \frac{\partial L}{\partial \dot{q}_1} = \frac{\partial L}{\partial \dot{x}} = m[\dot{x} - e\dot{\theta} \cos \theta] \]
\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) = m\ddot{x} - me\ddot{\theta} + me\dot{\theta}^2 \sin \theta \]
\[ \frac{\partial L}{\partial q_1} = \frac{\partial L}{\partial x} = -k_1 x \]

Now use the Lagrange equation to get:
\[ m\ddot{x} - me\ddot{\theta} \cos \theta + em\dot{\theta}^2 \sin \theta + k_1 x = 0 \]
Likewise differentiation with respect to \( q_2 = \theta \) yields:
\[ J\ddot{\theta} + me \cos \theta \ddot{x} + me^2 \cos^2 \theta \dddot{\theta} - me^2 \dot{\theta}^2 \sin \theta \cos \theta + k_2 \theta = 0 \]
Next Linearize and write in matrix form

Using the small angle approximations:
\[ \sin \theta \rightarrow \theta \quad \cos \theta \rightarrow 1 \]

In matrix form this becomes:
\[
\begin{bmatrix}
    m & -me \\
    -me & me^2 + J
\end{bmatrix}
\begin{bmatrix}
    \ddot{x}(t) \\
    \ddot{\theta}(t)
\end{bmatrix}
+ \begin{bmatrix}
    k_1 & 0 \\
    0 & k_2
\end{bmatrix}
\begin{bmatrix}
    x(t) \\
    \theta(t)
\end{bmatrix}
= \begin{bmatrix}
    0 \\
    0
\end{bmatrix}
\]

Note that this is a dynamically coupled system.
Next consider the Single Spring-Mass System and compute the equation of motion using the Lagrangian approach

\[ T = \frac{1}{2} m\dot{x}^2, \quad U = \frac{1}{2} kx^2 \]

\[ L = T - U = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2 \]

\[ \frac{\partial L}{\partial \dot{x}} = m\ddot{x}, \quad \frac{\partial L}{\partial x} = -kx \]

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0 \Rightarrow m\ddot{x} + kx = 0 \]