Some Review: Window 4.2

Orthonormal Vectors
similar to the unit vectors of statics and dynamics

\( \mathbf{x}_1 \) and \( \mathbf{x}_2 \) are both normal if \( \mathbf{x}_1^T \mathbf{x}_1 = 1 \) and \( \mathbf{x}_2^T \mathbf{x}_2 = 1 \)

and are orthogonal if \( \mathbf{x}_1^T \mathbf{x}_2 = 0 \)

This is abbreviated by

\[
\mathbf{x}_i^T \mathbf{x}_j = \delta_{ij} = \begin{cases} 
0 & \text{if } i \neq j \\
1 & \text{if } i = j 
\end{cases}
\]

A set of \( n \) vectors \( \mathbf{x}_i \) are set to be orthonormal if

\[
\mathbf{x}_i^T \mathbf{x}_j = \delta_{ij}
\]

for all values of \( i \) and \( j \).
4.3 - Modal Analysis

- Physical coordinates are not always the easiest to work in
- Eigenvectors provide a convenient transformation to modal coordinates
  - Modal coordinates are linear combination of physical coordinates
  - Say we have physical coordinates \( x \) and want to transform to some other coordinates \( u \)

\[
\begin{align*}
\mathbf{u}_1 &= x_1 + 3x_2 \\
\mathbf{u}_2 &= x_1 - 3x_2
\end{align*}
\]

\[
\begin{bmatrix}
\mathbf{u}_1 \\
\mathbf{u}_2
\end{bmatrix}
= \begin{bmatrix}
1 & 3 \\
1 & -3
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]
Review of the Eigenvalue Problem

Start with \( M\ddot{x}(t) + Kx = 0 \) where \( x \) is a vector and \( M \) and \( K \) are matrices. Assume initial conditions \( x_0 \) and \( \dot{x}_0 \). Rewrite as

\[
M^{\frac{1}{2}} M^{\frac{1}{2}} \ddot{x} + Kx = 0
\]

and let

\[
M^{\frac{1}{2}} x = q \implies x = M^{-\frac{1}{2}} q \quad \text{(coord. trans. #1)}
\]
Eigenproblem (cont)

Premultiply by $M^{-\frac{1}{2}}$ to get

$$
M^{-\frac{1}{2}} M^\frac{1}{2} \ddot{q} + M^{-\frac{1}{2}} K M^{-\frac{1}{2}} q = \ddot{q} + \tilde{K} q = 0
$$

(4.55)

- Now we have a symmetric, real matrix
- *Guarantees* real eigenvalues and distinct, mutually orthogonal eigenvectors
Eigenvectors = Mode Shapes?

Mode shapes are solutions to $M \omega^2 u = Ku$ in physical coordinates. Eigenvetors are characteristics of matrices. The two are related by a simple transformation, but they are not synonymous.
The eigenvectors of the symmetric PD matrix $\tilde{K}$ are orthonormal, i.e., $P^T P = I$. Are the mode shapes orthonormal? Using the transformation $x = M^{-\frac{1}{2}} q$, the mode shapes $U = M^{-\frac{1}{2}} P \Rightarrow P = M^{\frac{1}{2}} U$. Now, $P^T P = U^T M^{\frac{1}{2}} M^{\frac{1}{2}} U = U^T M U = I$. Thus, the mode shapes are orthogonal only w.r.t. the mass matrix.

Similarly, $U^T K U = P^T \underbrace{M^{-\frac{1}{2}} K M^{-\frac{1}{2}}}_{\tilde{K}} P = \Lambda$
The Matrix of eigenvectors can be used to decouple the equations of motion
If $P$ orthonormal (unitary), $P^T P = I \implies P^T = P^{-1}$
Thus, $P^T \tilde{K} P = \Lambda = \text{diagonal matrix of eigenvalues}$.
Back to $\ddot{q} + \tilde{K}q = 0$. Make the additional coordinate transformation $q = Pr$ and premultiply by $P^T$

$$P^T P \ddot{r} + P^T \tilde{K} Pr = I \ddot{r} + \Lambda r = 0 \quad (4.59)$$

- Now we have decoupled the EOM i.e., we have $n$ independent 2nd-order systems in modal coordinates $r(t)$
Writing out equation (4.59) yields

\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} \begin{bmatrix} 
\ddot{r}_1(t) \\
\ddot{r}_2(t)
\end{bmatrix} + \begin{bmatrix}
\omega_1^2 & 0 \\
0 & \omega_2^2
\end{bmatrix} \begin{bmatrix} 
r_1(t) \\
r_2(t)
\end{bmatrix} = \begin{bmatrix} 
0 \\
0
\end{bmatrix}
\] (4.60)

\[
\ddot{r}_1(t) + \omega_1^2 r_1(t) = 0 \quad (4.62)
\]
\[
\ddot{r}_2(t) + \omega_2^2 r_2(t) = 0 \quad (4.63)
\]

We must also transform the initial conditions

\[
\begin{bmatrix}
r_0 \\
\dot{r}_0
\end{bmatrix} = \begin{bmatrix}
r_1(0) \\
r_2(0)
\end{bmatrix} = \begin{bmatrix}
r_{10} \\
r_{20}
\end{bmatrix} = P^T q(0) = P^T M^{1/2} x(0) \quad (4.64)
\]

\[
\begin{bmatrix}
\dot{r}_0 \\
\ddot{r}_0
\end{bmatrix} = \begin{bmatrix}
\dot{r}_1(0) \\
\dot{r}_2(0)
\end{bmatrix} = \begin{bmatrix}
\dot{r}_{10} \\
\dot{r}_{20}
\end{bmatrix} = P^T \dot{q}(0) = P^T M^{1/2} \dot{x}(0) \quad (4.65)
\]
This transformation takes the problem from couple equations in the physical coordinate system in to decoupled equations in the modal coordinates.

\[
x = M^{-1} P r
\]

Physical Coordinates. Coupled equations

Modal Coordinates. Uncoupled equations

Figure 4.5
Modal Transforms to SDOF

- The modal transformation \( P^T M^{1/2} \) transforms our 2 DOF into 2 SDOF systems.
- This allows us to solve the two decoupled SDOF systems independently using the methods of chapter 1.
- Then we can recombine using the inverse transformation to obtain the solution in terms of the physical coordinates.
The free response is calculated for each mode independently using the formulas from chapter 1:

$$r_i(t) = \frac{\dot{r}_{i0}}{\omega_i} \sin \omega_i t + r_{i0} \cos \omega_i t, \quad i = 1, 2$$

or (see Window 4.3 for a reminder)

$$r_i(t) = \sqrt{r_{i0}^2 + \left(\frac{\dot{r}_{i0}}{\omega_i}\right)^2} \sin(\omega_i t + \tan^{-1}\frac{\omega_i r_{i0}}{\dot{r}_{i0}}), \quad i = 1, 2$$

Note, the above assumes neither frequency is zero
Once the solution in modal coordinates is determined \((r_i)\) then the response in Physical Coordinates is computed:

- With \(n\) DOFs these transformations are:

\[
\begin{bmatrix}
\mathbf{x}(t) \\
\mathbf{r}(t)
\end{bmatrix}
= 
\mathbf{S}
\begin{bmatrix}
\mathbf{r}(t)
\end{bmatrix}
\]

\[
\mathbf{S} = \mathbf{M}^{-\frac{1}{2}} \mathbf{P}
\]

(\text{where } n = 2 \text{ in the previous slides})
Steps in solving via modal analysis (Window 4.4)

1. Calculate $M^{-1/2}$.
2. Calculate $\tilde{K} = M^{-1/2} KM^{-1/2}$, the mass normalized stiffness matrix.
3. Calculate the symmetric eigenvalue problem for $\tilde{K}$ to get $\omega^2_i$ and $v_i$.
4. Normalize $v_i$ and form the matrix $P = [v_1 \quad v_2]$.
5. Calculate $S = M^{-1/2} P$ and $S^{-1} = P^T M^{1/2}$.
6. Calculate the modal initial conditions: $r(0) = S^{-1}x_0$, $\dot{r}(0) = S^{-1}\dot{x}_0$.
7. Substitute the components of $r(0)$ and $\dot{r}(0)$ into equations (4.66) and (4.67) to get the solution in modal coordinate $r(t)$.
8. Multiply $r(t)$ by $S$ to get the solution $x(t) = Sr(t)$.

Note that $S$ is the matrix of mode shapes and $P$ is the matrix of eigenvectors.
Example 4.3.1 via MATLAB (see text for hand calculations)

\[ M = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}, \quad K = \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix}, \quad x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \dot{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

• Follow steps in Window 4.4 (pg 300)

1) Calculate \( M^{-1/2} \)
2) Calculate \( \tilde{K} = M^{-1/2}K M^{-1/2} \)

\[
\begin{align*}
\text{» Minv2} &= \text{inv(sqrt(M))} \\
\text{Minv2} &= \\
&= \begin{bmatrix} 0.3333 & 0 \\ 0 & 1.0000 \end{bmatrix} \\
\text{» Kt} &= \text{Minv2*K*Minv2} \\
\text{Kt} &= \\
&= \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}
\end{align*}
\]
Example 4.3.1 solved using MATLAB as a calculator

% 3) Calculate the symmetric eigenvalue problem for K tilde
[P,D] = eig(Kt);
[lambda,I]=sort(diag(D)); % just sorts smallest to largest
P=P(:,I); % reorder eigenvectors to match eigenvalues

»lambda =
    2
    4
P =
   -0.7071   -0.7071
   -0.7071    0.7071
Example 4.3.1 (cont)

% 4) Calculate $S = M^{-1/2} \ast P$ and $S_{\text{inv}} = P^T \ast M^{1/2}$
S = Minv2 * P;
Sinv = inv(S);

% 5) Calculate the modal initial conditions
r0 = Sinv * x0;
rdot0 = Sinv * v0;
% 6) Find the free response in modal coordinates

tmax = 10;
numt = 1000;
t = linspace(0,tmax,numt);
[T,W]=meshgrid(t,lambda.^(1/2));

% Use Tony's trick
R0 = r0(:,ones(numt,1));
RDOT0 = rdot0(:,ones(numt,1));

r = RDOT0./W.*sin(W.*T) + R0.*cos(W.*T);

% 7) Transform back to physical space
x = S*r;
Example 4.3.1 (cont)

% Plot results
figure

subplot(2,1,1)
plot(t,r(1,:),'-',t,r(2,:),'--')
title('free response in modal coordinates')
xlabel('time (sec)')
legend('r_1','r_2')

subplot(2,1,2)
plot(t,x(1,:),'-',t,x(2,:),'--')
title('free response in physical coordinates')
xlabel('time (sec)')
legend('x_1','x_2')
Modal and Physical Responses

Modal Coordinates: Independent oscillators

\[ \lambda_1 = 2 \Rightarrow \omega_1 = \sqrt{2} \]
\[ \Rightarrow T_1 = \sqrt{2}\pi = 4.44 \text{ sec} \]
\[ \lambda_2 = 4 \Rightarrow \omega_1 = 2 \]
\[ \Rightarrow T_2 = \pi \text{ sec} \]

Physical Coordinates: Coupled oscillators

Note IC
Section 4.4 More than 2 Degrees of Freedom

Extending previous section to any number of degrees of freedom
A FBD of the system of figure 4.8 yields the \( n \) equations of motion of the form:

\[
    m_i + k_i (x_i - x_{i-1}) - k_{i+1} (x_{i-1} - x_i) = 0, \quad i = 1, 2, 3\ldots n \quad (4.83)
\]

Writing all \( n \) of these equations and casting them in matrix form yields:

\[
    M\ddot{x}(t) + Kx(t) = 0, \quad (4.80)
\]

where:
the relevant matrices and vectors are:

\[
M = \begin{bmatrix}
m_1 & 0 & L & 0 \\
0 & m_2 & L & 0 \\
M & M & O & M \\
0 & 0 & L & m_n
\end{bmatrix}, \quad K = \begin{bmatrix}
k_1 + k_2 & -k_2 & 0 & L & 0 \\
-k_2 & k_2 + k_3 & -k_3 & 0 \\
0 & -k_3 & O & O & M \\
M & O & k_{n-1} + k_n & -k_n \\
0 & 0 & L & -k_n & k_n
\end{bmatrix}
\] (4.83)

\[
x(t) = \begin{bmatrix}
x_1(t) \\
x_2(t) \\
\vdots \\
x_n(t)
\end{bmatrix}, \quad \ddot{x}(t) = \begin{bmatrix}
\ddot{x}_1(t) \\
\ddot{x}_2(t) \\
\vdots \\
\ddot{x}_n(t)
\end{bmatrix}
\]
For such systems as figure 4.7 and 4.8 the process stays the same...just more modal equations result:

Process stays the same as section 4.3

\[
\ddot{r}_1(t) + \omega_1^2 r_1(t) = 0
\]

\[
\ddot{r}_2(t) + \omega_2^2 r_2(t) = 0
\]

\[
\ddot{r}_3(t) + \omega_3^2 r_3(t) = 0
\]

\[\vdots\]

\[
\ddot{r}_n(t) + \omega_n^2 r_n(t) = 0
\]

Just get more modal equations, one for each degree of freedom (\(n\) is the number of dof)

See example 4.4.2 for details
The Mode Summation Approach

• Based on the idea that any possible time response is just a linear combination of the eigenvectors

Starting with \( \ddot{q}(t) + \tilde{K}q(t) = 0 \) \hspace{1cm} (4.88)

let \( q(t) = \sum_{i=1}^{n} q_i(t) = \sum_{i=1}^{n} \left( a_i e^{-j\sqrt{\lambda_i} t} + b_i e^{j\sqrt{\lambda_i} t} \right) v_i \)

\( \Rightarrow \) two linearly independent solutions for each term.

can also write this as \( q(t) = \sum_{i=1}^{n} d_i \sin(\omega_i t + \phi_i) v_i \) \hspace{1cm} (4.92)
Mode Summation Approach (cont)

Find the constants $d_i$ and $\phi_i$ from the I.C.

$$
q(0) = \sum_{i=1}^{n} d_i \sin \phi_i \mathbf{v}_i \quad \text{and} \quad \dot{q}(0) = \sum_{i=1}^{n} d_i \omega_i \cos \phi_i \mathbf{v}_i
$$

Assuming eigenvectors normalized such that $\mathbf{v}_j^T \mathbf{v}_i = \delta_{ij}$

$$
\mathbf{v}_j^T q(0) = \mathbf{v}_j^T \sum_{i=1}^{n} d_i \sin \phi_i \mathbf{v}_i = \sum_{i=1}^{n} d_i \sin \phi_i (\mathbf{v}_j^T \mathbf{v}_i) = d_j \sin \phi_j
$$

Similarly for the initial velocities, $\mathbf{v}_j^T \dot{q}(0) = d_j \omega_j \cos \phi_j$
Mode Summation Approach (cont)

Solve for \( d_i \) and \( \phi_i \) from the two IC equations

\[
d_i = \frac{v_i^T q(0)}{\sin \phi_i} \quad \text{and} \quad \phi_i = \tan^{-1} \left( \frac{\omega_i v_i^T q(0)}{v_i^T \dot{q}(0)} \right)
\]

IMPORTANT NOTE about \( q(0) = 0 \)

if you just crank it through the above expressions you might conclude that \( d_i = 0 \), i.e., the trivial soln.

Be careful with \( \dot{q}(0) = 0 \) as well.
Mode Summation Approach for zero initial displacement

If \( q(0) = 0 \), the return to

\[
q(0) = \sum_{i=1}^{n} d_i \sin \phi_i \mathbf{v}_i
\]

and realize that \( \phi_i = 0 \) instead of \( d_i = 0 \).

The compute \( d_i \) from the velocity expression

\[
\mathbf{v}_i^T \dot{q}(0) = \omega_i d_i \cos \phi_i
\]
Mode Summation Approach with rigid body modes \((\omega_1 = 0)\)

if \(\lambda_1 = 0\),

\[
q_1(t) = \left( a_1 e^{-j\sqrt{\lambda_1}t} + b_1 e^{j\sqrt{\lambda_1}t} \right) v_i = (a_1 + b_1)v_i
\]

does not give two linearly independent solutions. Now we must use the expansion

\[
q(t) = (a_1 + b_1 t) v_1 + \sum_{i=2}^{n} \left( a_i e^{-j\sqrt{\lambda_i}t} + b_i e^{j\sqrt{\lambda_i}t} \right) v_i
\]

and adjust calculation of the constants from the initial conditions accordingly.

Note that the underline term is a translational motion
Example 4.3.1 solved by the mode summation method

From before, we have \( M^{1/2} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \) and \( V = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \)

Appropriate IC are \( q_0 = M^{1/2} x_0 = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \), \( \dot{q}_0 = M^{1/2} v_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \)

\[
\phi_i = \tan^{-1} \frac{\omega_i v_i^T q(0)}{v_i^T \dot{q}(0)} = \tan^{-1} \frac{\omega_i v_i^T q(0)}{0} \Rightarrow \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} \frac{\pi}{2} \\ -\frac{\pi}{2} \end{bmatrix}
\]

\[
d_i = \frac{v_i^T q(0)}{\sin \phi_i} \Rightarrow \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 3\sqrt{2}/2 \\ 3\sqrt{2}/2 \end{bmatrix}
\]
Example 4.3.1 constructing the summation of modes

\[
\begin{bmatrix}
q_1(t) \\
q_2(t)
\end{bmatrix}
= \frac{3\sqrt{2}}{2} \sin\left(\sqrt{2}t - \frac{\pi}{2}\right) \frac{1}{\sqrt{2}} \begin{bmatrix}
-1 \\
-1
\end{bmatrix} + \frac{3\sqrt{2}}{2} \sin\left(2t - \frac{\pi}{2}\right) \frac{1}{\sqrt{2}} \begin{bmatrix}
1 \\
1
\end{bmatrix}
\]

the first mode

the second mode

Transforming back to the physical coordinates yields:

\[
x(t) = M^{-1/2} q = \frac{3\sqrt{2}}{2} \sin\left(\sqrt{2}t - \frac{\pi}{2}\right) \frac{1}{\sqrt{2}} \begin{bmatrix}
\frac{1}{3} & 0 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
-1 \\
-1
\end{bmatrix} + \frac{3\sqrt{2}}{2} \sin\left(2t - \frac{\pi}{2}\right) \frac{1}{\sqrt{2}} \begin{bmatrix}
\frac{1}{3} & 0 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
1 \\
1
\end{bmatrix}
\]

\[
= \frac{3\sqrt{2}}{2} \sin\left(\sqrt{2}t - \frac{\pi}{2}\right) \frac{1}{3\sqrt{2}} \begin{bmatrix}
-1 \\
-1
\end{bmatrix} + \frac{3\sqrt{2}}{2} \sin\left(2t - \frac{\pi}{2}\right) \frac{1}{3\sqrt{2}} \begin{bmatrix}
1 \\
1
\end{bmatrix}
\]

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30/53

Mechanical Engineering at Virginia Tech
Example 4.3.1  a comparison of the two solution methods shows they yield identical results
Steps for Computing the Response By Mode Summation

1. Write the equations of motion in matrix form, identify $M$ and $K$
2. Calculate $M^{-1/2}$ (or $L$)
3. Calculate $\tilde{K} = M^{-\frac{1}{2}}KM^{-\frac{1}{2}}$
4. Compute the eigenvalue problem for the matrix $\tilde{K}$ and get $\omega_i^2$ and $v_i$
5. Transform the initial conditions to $q(t)$
   
   \[ q(0) = M^{\frac{1}{2}}x(0) \text{ and } \quad \dot{q}(0) = M^{\frac{1}{2}}\dot{x}(0) \]
Summary of Mode Summation Continued

6. Calculate the modal expansion coefficients and phase constants

\[ \phi_i = \tan^{-1}\left( \frac{\omega_i v_i^T q(0)}{v_i^T \dot{q}(0)} \right), \quad d_i = \frac{v_i^T q(0)}{\sin \phi_i} \]

7. Assemble the time response for \( q \)

\[ q(t) = \sum_{i=1}^{n} d_i \sin(\omega_i t + \phi_i) v_i \]

8. Transform the solution to physical coordinates

\[ x(t) = M^{-1/2} q(t) = \sum_{i=1}^{n} d_i \sin(\omega_i t + \phi_i) u_i \]
Nodes of a Mode Shape

- Examination of the mode shapes in Example 4.4.3 shows that the third entry of the second mode shape is zero!
- Zero elements in a mode shape are called nodes.
- A node of a mode means there is no motion of the mass or (coordinate) corresponding to that entry at the frequency associated with that mode.
The second mode shape of Example 4.4.3 has a node

- Note that for more than 2 DOF, a mode shape may have a zero valued entry
- This is called a node of a mode.

\[ u_2 = \begin{bmatrix} 0.2887 \\ 0.2887 \\ 0 \\ -0.2887 \end{bmatrix} \]

They make great mounting points in machines
A rigid body mode is the mode associated with a zero frequency

Fig 4.12

- Note that the system in Fig 4.12 is not constrained and can move as a rigid body
- Physically if this system is displaced we would expect it to move off the page whilst the two masses oscillate back and forth
Example 4.4.4 Rigid body motion

The free body diagram of figure 4.12 yields

\[ m_1 \ddot{x}_1 = k(x_2 - x_1) \quad \text{and} \quad m_2 \ddot{x}_2 = -k(x_2 - x_1) \]

\[
\begin{bmatrix}
  m_1 & 0 \\
  0 & m_2 
\end{bmatrix}
\begin{bmatrix}
  \ddot{x}_1 \\
  \ddot{x}_2 
\end{bmatrix}
+ k
\begin{bmatrix}
  1 & -1 \\
  -1 & 1 
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2 
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

Solve for the free response given:

\[ m_1 = 1 \text{ kg}, \quad m_2 = 4 \text{ kg}, \quad k = 400 \text{ N} \]

subject to

\[ x_0 = \begin{bmatrix} 0.01 \\ 0 \end{bmatrix} \text{ m } \quad \text{and} \quad v_0 = 0 \]
Following the steps of Window 4.4

1. \( M^{-1/2} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \)

2. \( \tilde{K} = M^{-1/2} KM^{-1/2} = 400 \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 400 & -200 \\ -200 & 100 \end{bmatrix} \)

3. \( \det (\tilde{K} - \lambda I) = 100 \det \left( \begin{bmatrix} 4 - \lambda & -2 \\ -2 & 1 - \lambda \end{bmatrix} \right) = 100 (\lambda^2 - 5\lambda) = 0 \)

\( \Rightarrow \lambda_1 = 0 \) and \( \lambda_2 = 5 \) \( \Rightarrow \omega_1 = 0, \quad \omega_2 = 2.236 \text{ rad/s} \)

Indicates a rigid body motion
Now calculate the eigenvectors and note in particular that they cannot be zero even if the eigenvalue is zero

\[ \lambda = 0 \Rightarrow 100 \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 4v_{11} - 2v_{21} = 0 \]

\[ \Rightarrow v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ or after normalizing } \quad v_1 = \begin{bmatrix} 0.4472 \\ 0.8944 \end{bmatrix} \]

Likewise: \[ v_2 = \begin{bmatrix} -0.8944 \\ 0.4472 \end{bmatrix} \Rightarrow P = \begin{bmatrix} 0.4472 & -0.8944 \\ 0.8944 & 0.4472 \end{bmatrix} \]

As a check note that

\[ P^T P = I \quad \text{and} \quad P^T \tilde{K} P = \text{diag} [0 \quad 5] \]
5. Calculate the matrix of mode shapes

\[ S = M^{-1/2} P = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 0.4472 & -0.8944 \\ 0.8944 & 0.4472 \end{bmatrix} = \begin{bmatrix} 0.4472 & -0.8944 \\ 0.4472 & 0.2236 \end{bmatrix} \]

\[ \Rightarrow S^{-1} = \begin{bmatrix} 0.4472 & 1.7889 \\ -0.8944 & 0.8944 \end{bmatrix} \]

7. Calculate the modal initial conditions:

\[ \mathbf{r}(0) = S^{-1} \mathbf{x}_0 = \begin{bmatrix} 0.4472 & 1.7889 \\ -0.8944 & 0.8944 \end{bmatrix} \begin{bmatrix} 0.01 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.004472 \\ -0.008944 \end{bmatrix} \]

\[ \dot{\mathbf{r}}(0) = S^{-1} \dot{\mathbf{x}}_0 = 0 \]
7. Now compute the solution in modal coordinates and note what happens to the first mode.

Since $\omega_1 = 0$ the first modal equation is

$$\ddot{r}_1 + (0)r_1 = 0$$

$$\Rightarrow r_1(t) = a + bt \quad \text{Rigid body translation}$$

And the second modal equation is

$$\ddot{r}_2(t) + 5r_2(t) = 0$$

$$\Rightarrow r_2(t) = a_2 \cos \sqrt{5}t \quad \text{Oscillation}$$
Applying the modal initial conditions to these two solution forms yields:

\[ r_1(0) = a = 0.004472 \]
\[ \dot{r}_1(0) = b = 0.0 \]
\[ \Rightarrow r_1(t) = 0.0042 \]

as in the past problems the initial conditions for \( r_2 \) yield

\[ r_2(t) = -0.0089 \cos \sqrt{5}t \]

\[ \Rightarrow \mathbf{r}(t) = \begin{bmatrix} 0.0042 \\ -0.0089 \cos \sqrt{5}t \end{bmatrix} \]
8. Transform the modal solution to the physical coordinate system

\[
x(t) = Sr(t) = \begin{bmatrix}
0.4472 & -0.8944 \\
0.4472 & 0.2236
\end{bmatrix}\begin{bmatrix}
0.0045 \\
-0.0089 \cos \sqrt{5}t
\end{bmatrix}
\]

\[
\Rightarrow x(t) = \begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix} = \begin{bmatrix}
2.012 + 7.60 \cos \sqrt{5}t \\
2.012 - 1.990 \cos \sqrt{5}t
\end{bmatrix} \times 10^{-3} \text{ m}
\]

Each mass is moved a constant distance and then oscillates at a single frequency.
Order the frequencies

• It is convention to call the lowest frequency $\omega_1$ so that $\omega_1 \leq \omega_2 \leq \omega_3 < \ldots$
• Order the modes (or eigenvectors) accordingly
• It really does not make a difference in computing the time response
• However:
  – When we measuring frequencies, they appear lowest to highest
  – Physically the frequencies respond with the highest energy in the lowest mode (important in flutter calculations, run up in rotating machines, etc.)
The system of Example 4.1.5 solved by Mode Summation

From Example 4.1.6 we have:

\[ \omega_1 = \sqrt{2}, \quad u_1 = \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}, \quad \omega_2 = 2, \quad u_2 = \begin{bmatrix} -1/3 \\ 1 \end{bmatrix} \]

Use the following initial conditions and note that only one mode should be excited (why?)

\[ x(0) = \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}, \quad \dot{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]
Transform coordinates

\[ M = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow M^{\frac{1}{2}} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad M^{-\frac{1}{2}} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix} \]

Thus the initial conditions become

\[ q(0) = M^{\frac{1}{2}} x(0) = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \]

\[ \dot{q}(0) = M^{\frac{1}{2}} \dot{x}(0) = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]
Transform Mode Shapes to Eigenvectors

\[ v_1 = M^{\frac{1}{2}}u_1 = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \]

\[ v_2 = M^{\frac{1}{2}}u_2 = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \]

Note that unlike the mode shapes, the eigenvectors are orthogonal:

\[ v_1^T v_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = 0, \text{ but } u_1^T u_2 = \begin{bmatrix} \frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix} = \frac{2}{3} \neq 0 \]

Normalizing yields:

\[ v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \]
From Equation (4.92):

\[ q(t) = \sum_{i=1}^{2} d_i \sin(\omega_i t + \phi_i) v_i \Rightarrow \dot{q}(t) = \sum_{i=1}^{2} d_i \omega_i \cos(\omega_i t + \phi_i) v_i \]

Set \( t=0 \) and multiply by \( v_1 \):

\[ \dot{q}(0) = \sum_{i=1}^{2} d_i \omega_i \cos \phi_i v_i \]

\[ \Rightarrow v_1^T \begin{bmatrix} 0 \\ 0 \end{bmatrix} = d_1 \sqrt{2} \cos \phi_1 v_1^T \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + d_2 \frac{2}{2} \cos \phi_2 v_1^T \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \]

\[ \Rightarrow 0 = d_1 \cos \phi_1 \Rightarrow \phi_1 = \pi / 2 \]

Or directly from Eq. (4.97)
From the initial displacement:

\[ d_1 = \frac{\mathbf{v}_1^T \mathbf{q}(0)}{\sin(\pi / 2)} = \begin{bmatrix} 1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{2}{\sqrt{2}} \]

\[ d_2 = \frac{\mathbf{v}_2^T \mathbf{q}(0)}{\sin(\pi / 2)} = \begin{bmatrix} -1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0 \]  

Thus

\[ \mathbf{q}(t) = \sum_{i=1}^{2} d_i \sin(\omega_i t + \phi_i) \mathbf{v}_i \]

\[ = \sqrt{2} \cos(\sqrt{2}t) \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \cos(\sqrt{2}t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \]
Transforming Back to Physical Coordinates:

\[ x(t) = M^{-\frac{1}{2}}q(t) = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix} \cos(\sqrt{2}t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \]

\[ = \begin{bmatrix} \frac{1}{3} \cos \sqrt{2}t \\ \\
\cos \sqrt{2}t \end{bmatrix} \]

\[ \Rightarrow x_1(t) = \frac{1}{3} \cos \sqrt{2}t \quad \text{and} \quad x_2(t) = \cos \sqrt{2}t \]

So, the initial conditions generated motion only in the first mode (as expected)
Alternate Path to Symmetric Single-Matrix Eigenproblem

• Square root of matrix conceptually easy, but computationally expensive

\[ M^{-\frac{1}{2}} M^{\frac{1}{2}} \ddot{\mathbf{q}} + M^{-\frac{1}{2}} K M^{-\frac{1}{2}} \mathbf{q} = \ddot{\mathbf{q}} + \tilde{\mathbf{K}} \mathbf{q} = 0 \]

• More efficient to decompose \( M \) into product of upper and lower triangular matrices (Cholesky decomposition)
Cholesky Decomposition

Let $M = U^T U$ where $U$ is upper triangular
Introduce the coordinate transformation

$$Ux = q \implies x = U^{-1}q \implies U^T U \ddot{x} + K \dot{q} = 0$$

premultiply by $U^{-T}$ to get

$$I \ddot{q} + U^{-T} K U^{-1} q = \ddot{q} + \tilde{K} q = 0$$

note that:

$$\left[U^{-T} K U^{-1}\right]^T = \left[U^{-1}\right]^T K^T \left[U^{-T}\right]^T = U^{-T} K U^{-1}$$
Cholesky (cont)

• Is this really faster? Let’s ask MATLAB

\[ M = M^{\frac{1}{2}} M^{\frac{1}{2}} \]
\[ M = U^T U \]

```matlab
» M = [9 0 ; 0 1];
» flops(0); sqrtm(M); flops
ans = 65

» M = [9 0 ; 0 1];
» flops(0); chol(M); flops
ans = 5
```

• `sqrtm` requires a singular value decomposition (SVD), whereas Cholesky requires only simple operations

Note that \( M^{\frac{1}{2}} = U \) for diagonal \( M \)