3.5 Random Vibrations

- So far our excitations have been harmonic, periodic, or at least known in advance.
- These are examples of deterministic excitations, i.e., known in advance for all time.
  - That is given \( t \) we can predict the value of \( F(t) \) exactly.
- Responses are deterministic as well.
- Many physical excitations are nondeterministic, or random, i.e., can’t write explicit time descriptions.
  - Rockets
  - Earthquakes
  - Aerodynamic forces
  - Rough roads and seas.
- The responses \( x(t) \) are also nondeterministic.
Random Vibrations

- *Stationary* signals are those whose statistical properties do not vary over time
- Functions are described in terms of probabilities
  - Mean values*
  - Standard deviations
- Random outputs related to random input via system transfer function

*ie given $t$ we do not know $x(t)$ exactly, but rather we only know statistical properties of the response such as the average value
Autocorrelation function and power spectral density

The autocorrelation function describes how a signal is changing in time or how correlated the signal is at two different points in time.

\[ R_{xx}(t) = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} x(t) x(t + \tau) d\tau \]

The Power Spectral Density describes the power in a signal as a function of frequency and is the Fourier transform of the autocorrelation function.

\[ S_{xx}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-j\omega \tau} d\tau \]

FFT is a new way for calculating Fourier Transforms
Examples of signals

HARMONIC

- \[ x(t) \]
- \[ A \]
- \[ -A \]
- \[ A^2/2 \]
- \[ -A^2/2 \]
- \[ R_{xx}(\tau) \]
- \[ T \]
- \[ 1/T \]
- \[ S_{xx}(\omega) \]

RANDOM

- \[ x(t) \]
- \[ A_{rms} \]
- \[ A^2_{rms} \]
- \[ R_{xx}(\tau) \]
- \[ \tau \text{ time shift} \]
- \[ 1/T \]
- \[ S_{xx}(\omega) \]

Signal

Autocorrelation

Power Spectral Density

Frequency (Hz)

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More Definitions

Average:

\[ \bar{x} = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} x(t) dt \tag{3.47} \]

Mean-square:

\[ x^2 = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} x^2(t) dt \tag{3.48} \]

rms:

\[ x_{\text{rms}} = \sqrt{x^2} = \sqrt{\lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} x^2(t) dt} \tag{3.49} \]
Expected Value
(or ensemble average)

The expected value is given by

\[ E[x(t)] = \lim_{T \to \infty} \int_{0}^{T} \frac{x(t)}{T} dt = \bar{x} \quad (3.63) \]

The Probability Density Function, \( p(x) \), is the probability that \( x \) lies in a given interval (e.g. Gaussian Distribution)

The expected value is also given by

\[ E[x] = \int_{-\infty}^{\infty} xp(x)dx \quad (3.64) \]
Recall the Basic Relationships for Transforms:

Recall for SDOF:

transfer function: \( G(s) = \frac{1}{ms^2 + cs + k} \)

frequency response function: \( G(j\omega) = H(\omega) = \frac{1}{k - m\omega^2 + c\omega j} \)

unit impulse response function: \( h(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_d t} \sin \omega_d t \)

\[ L[h(t)] = \frac{1}{ms^2 + cs + k} = G(s) \]

And the Fourier Transform of \( h(t) \) is \( H(\omega) \)
What can you predict?

The response of SDOF with $f(t)$ as input:

**Deterministic Input:**

$$X(s) = G(s)F(s)$$

$$x(t) = \int_0^t h(t-\tau) f(\tau)d\tau$$

**Random Input:**

$$S_{xx}(\omega) = |H(\omega)|^2 S_{ff}(\omega)$$

$$E[x^2] = \int_{-\infty}^{\infty} |H(\omega)|^2 S_{ff}(\omega)d\omega$$

In a Lab, the PSD function of a random input and the output can be measured simply in one experiment. So the FRF can be computed as their ratio by a single test, instead of performing several tests at various constant frequencies.

Here we get an exact time record of the output given an exact record of the input.

Here we get an expected value of the output given a statistical record of the input.
Example 3.5.1 PSD Calculation

Consider \( m\ddot{x} + c\dot{x} + kx = F(t) \), where the PSD of \( F(t) \) is constant \( S_0 \)

The corresponding frequency response function is:

\[
H(\omega) = \frac{1}{k - m\omega^2 + c\omega j} \quad (2.59)
\]

\[
\Rightarrow |H(\omega)|^2 = \left| \frac{1}{k - m\omega^2 + c\omega j} \right|^2 = \frac{1}{(k - m\omega^2)^2 + c^2\omega^2}
\]

From equation (3.62) the PSD of the response becomes:

\[
S_{xx} = |H(\omega)|^2 \quad S_{ff} = \frac{S_0}{(k - m\omega^2)^2 + (c\omega)^2}
\]
Example 3.5.2  Mean Square Calculation

Consider the system of Example 3.5.1 and compute:

\[ E \left[ x^2 \right] = S_0 \int_{-\infty}^{\infty} \left| \frac{1}{k - m\omega_n^2 + c\omega j} \right|^2 d\omega \]

\[ = S_0 \frac{\pi m}{kcm} = \frac{\pi S_0}{kc} \]

Here the evaluation of the integral is from a tabulated value. See equation (3.70).
Section 3.6 Shock Spectrum

Arbitrary forms of shock are probable (earthquakes, …)

The spectrum of a given shock is a plot of the maximum response quantity \( (x) \) against the ratio of the forcing characteristic (such as rise time) to the natural period.

Maximum response gives maximum stress.

\[
x(t) = \int_0^t F(\tau) h(t - \tau) d\tau \quad (3.71)
\]
Using the convolution equation as a tool, compute the maximum value of the response

Recall the impulse response function undamped system:

\[ h(t - \tau) = \frac{1}{m\omega_n} \sin \omega_n (t - \tau) \quad (3.73) \]

\[ \Rightarrow \]

\[ x(t)_{\max} = \frac{1}{m\omega_n} \left| \int_0^t F(\tau) \sin(\omega_n (t - \tau)) d\tau \right|_{\max} \quad (3.74) \]

Such integrals usually have to be computed numerically.
Example 3.6.1 Compute the response spectrum for gradual application of a constant force $F_0$. Assume zero initial conditions

$$m\ddot{x}(t) + kx(t) = F(t)$$

$t_1=$infinity, means static loading

$$F(t) = F_1(t) + F_2(t)$$

$$F_1(t) = \frac{t}{t_1}F_0$$

time shift and negative, like half sine problem

$$F_2(t) = \begin{cases} 
0 & 0 < t < t_1 \\
-(\frac{t-t_1}{t_1})F_0 & t \geq t_1 
\end{cases}$$

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Split the solution into two parts and use the convolution integral

\[ x_1(t) = \frac{\omega_n}{k} \int_0^t \frac{F_0 \tau}{t_1} \sin \omega_n (t - \tau) d\tau = \frac{F_0}{k} \left( \frac{t}{t_1} - \frac{\sin \omega_n t}{\omega_n t_1} \right) \quad 0 < t < t_1 \]

(3.75)

For \( x_2 \) apply time shift \( t_1 \)

\[ x_2(t) = -\frac{F_0}{k} \left( \frac{t - t_1}{t_1} - \frac{\sin \omega_n (t - t_1)}{\omega_n t_1} \right), \quad t \geq t_1 \]

(3.76)

\[ x(t) = x_1(t) + x_2(t) = \frac{F_0}{k} \left( \frac{t}{t_1} - \frac{\sin \omega_n t}{\omega_n t_1} \right) - \frac{F_0}{k} \left( \frac{t - t_1}{t_1} - \frac{\sin \omega_n (t - t_1)}{\omega_n t_1} \right) \Phi(t - t_1) \]

(3.77)
Next find the maximum value of this response

To get max response, differentiate \( x(t) \).

In the case of a harmonic input (Chapter 2) we computed this by looking at the coefficient of the steady state response, giving rise to the Magnitude plots of figures 2.7, 2.8, 2.13.

Need to look at two cases 1) \( t < t_1 \) and 2) \( t \geq t_1 \)

For case 2) solve: (what about case 1? Its max is \( X_{\text{static}} \))

\[
\frac{d}{dt} \left[ \frac{F_0}{k \omega_n t_1} \left( \omega_n t_1 - \sin \omega_n t + \sin \omega_n (t - t_1) \right) \right] = 0 \Rightarrow
\]
Solve for $t$ at $x_{\text{max}}$, denoted $t_p$

$$\left. - \cos \omega_n t + \cos \omega_n (t - t_1) \right|_{t=t_p} = 0$$

$$\cos \omega_n t_p = \cos \omega_n t_p \cos \omega_n t_1 + \sin \omega_n t_p \sin \omega_n t_1$$

$$\Rightarrow \omega_n t_p = \tan^{-1} \left( \frac{1 - \cos \omega_n t_1}{\sin \omega_n t_1} \right)$$

$$= \sqrt{\sin^2 \omega_n t_1 + (1 - \cos \omega_n t_1)^2}$$

$$= \sqrt{2(1 - \cos \omega_n t_1)}$$

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From the triangle:

\[
\sin \omega_n t_p = -\sqrt{\frac{1}{2}(1 - \cos \omega_n t_1)}
\]

\[
\cos \omega_n t_p = -\frac{-\sin \omega_n t_1}{\sqrt{2(1 - \cos \omega_n t_1)}}
\]

Minus sq root taken as + gives a negative magnitude

Substitute into \(x(t_p)\) to get nondimensional \(X_{\text{max}}\):

\[
\frac{x_{\text{max}}}{F_0} = 1 + \frac{1}{\omega_n t_1} \sqrt{2(1 - \cos \omega_n t_1)}
\]

1\textsuperscript{st} term is static, 2\textsuperscript{nd} is dynamic. Plot versus:

\[
\frac{t_1}{T} = \frac{\omega_n t_1}{2 \pi}
\]

\(T\) is System period

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Response Spectrum

- Indicates how normalized max output changes as the input pulse width increases.
- Very much like a magnitude plot.
- Shows very small $t_1$ can increase the response significantly: impact, rather than smooth force application.
- The larger the rise time, the smaller the peaks.
- The maximum displacement is minimized if rise time is a multiple of natural period.
- Design by MiniMax idea.

$$X(t_1) := 1 + \frac{1}{\omega_n \cdot t_1} \cdot \sqrt{2 \cdot (1 - \cos(\omega_n \cdot t_1))}$$

$$X = \frac{x_{\text{max}} \cdot k}{F_0}$$
Comparison between impulse and harmonic inputs

**Impulse Input**

- **Transient Output**
- Max amplitude versus normalized pulse “frequency”

\[
\omega_n := 2 \cdot \pi \\
T := \frac{\omega_n}{2 \cdot \pi}
\]

\[
X(t_1) := 1 + \frac{1}{\omega_n \cdot t_1} \cdot \sqrt{2 \cdot \left(1 - \cos(\omega_n \cdot t_1)\right)}
\]

---

**Harmonic Input**

- **Harmonic Output**
- Max amplitude versus normalized driving frequency

\[
X(r, \zeta) := \frac{1}{\sqrt{(1 - r^2)^2 + (2 \cdot \zeta \cdot r)^2}}
\]

\[r := 0, 0.01 \ldots 2\]
Review of The Procedure for Shock Spectrum

1. Find $x(t)$ using convolution integral
2. Compute its time derivative
3. Set it equal to zero
4. Find the corresponding time
5. Evaluate the max possible value of $x$ (be careful about points where the function does not have derivative!!)
6. Plot it for different input shocks
3.7 Measurement via Transfer Functions

- Apply a sinusoidal input and measure the response
- Do this at small frequency steps
- The ratio of the Laplace transform of these to signals then gives an experiment transfer function of the system
Several different signals can be measured and these are named

<table>
<thead>
<tr>
<th>TABLE 3.2  TRANSFER FUNCTIONS USED IN VIBRATION MEASUREMENT</th>
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<td>Acceleration</td>
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<tr>
<td>Velocity</td>
</tr>
<tr>
<td>Displacement</td>
</tr>
</tbody>
</table>

receptance: \[ \frac{X(s)}{F(s)} = \frac{1}{ms^2 + cs + k} \] (3.86)

mobility: \[ \frac{sX(s)}{F(s)} = \frac{s}{ms^2 + cs + k} \] (3.87)

inertance: \[ \frac{s^2 X(s)}{F(s)} = \frac{s^2}{ms^2 + cs + k} \] (3.87)
The magnitude of the compliance transfer function yields information about the system parameters:

\[
|H(j\omega)| = \frac{1}{\sqrt{(k - m\omega)^2 + (c\omega)^2}} \quad (3.89)
\]

\[
|H(j\sqrt{\frac{k}{m}})| = \frac{1}{c\omega_n} \quad (3.90)
\]

\[
|H(0)| = \frac{1}{k} \quad (3.91)
\]
3.8 Stability

Stability is *defined* for the solution of free response case:

**Stable:** \( |x(t)| < M, \; \forall \; t > 0 \)

**Asymptotically Stable:** \( \lim_{t \to \infty} x(t) = 0 \)

**Unstable:**

if it is not stable or asymptotically stable
Recall these stability definitions for the free response:

- **Stable**: $x(t)$, $y(t)$, $z(t)$, $r(t)$

- **Asymptotically Stable**: $x(t)$, $y(t)$

- **Divergent instability**: $z(t)$

- **Flutter instability**: $r(t)$

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Stability for the forced response:

\[ m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F(t) \]

- Bounded Input-Bounded Output Stable
  \( x(t) \) bounded for ANY bounded \( F(t) \)

- Lagrange Stable with respect to \( F(t) \)
  If \( x(t) \) is bounded for THE given \( F(t) \)
Relationship between stability of the homogeneous system and the force response

- If $x_{\text{homo}}$ is Asymptotically stable then the forced response is BIBO stable (Bounded input, bounded output)
- If $x_{\text{homo}}$ is Stable then the forced response MAY be Lagrange Stable or Unstable
Stability for Harmonic Excitations

The solution to:

\[ m\ddot{x}(t) + kx(t) = F_0 \cos \omega t \]

is:

\[
x(t) = \frac{v_0}{\omega_n} \sin \omega_n t + \left( x_0 - \frac{f_0}{\omega^2 - \omega_n^2} \right) \cos \omega_n t + \frac{f_0}{\omega^2 - \omega_n^2} \cos \omega t
\]

As long as \( \omega_n \) is not equal to \( \omega \) this is Lagrange Stable, if the frequencies are equal it is Unstable.
For underdamped systems:

\[ m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F_0 \cos \omega t \]

\[ x_p(t) = \frac{f_0}{\sqrt{\left(\omega_n^2 - \omega^2\right)^2 + (2\zeta\omega_n\omega)^2}} \cos(\omega t - \tan^{-1}\left(\frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2}\right)) \]

Add homogeneous and particular to get total solution:

\[ x(t) = Ae^{-\zeta\omega_nt} \sin(\omega_d t + \phi) + X \cos(\omega t - \theta) \]

- homogeneous or transient solution
- particular or steady state solution

**Bounded Input-Bounded Output Stable**
Example 3.8.1

The equation of motion after a small angle approximation is given becomes:

\[ m\ell^2 \ddot{\theta}(t) + \frac{k\ell^2}{2} \theta(t) = mg \ell \theta(t) \]

\[ \Rightarrow m\ell^2 \ddot{\theta}(t) + \left( \frac{k\ell^2}{2} - mg \ell \right) \theta(t) = 0 \]

This will be stable if and only if the coefficient of \( \theta \) is positive or if \( k\ell > 2mg \)

- The system is thus Lagrange stable.
- Physically this tells us the spring must be large enough to overcome gravity.
3.9 Numerical Simulation of the response

- As before in Section 2.8 write equations of motion as state space equations
- The Euler integration is just

\[ \mathbf{x}(t_{i+1}) = \mathbf{x}(t_i) + A\mathbf{x}(t_i)\Delta t + \mathbf{F}(t_i)\Delta t \]
Example 3.9.1 with delay

Let the input force be a step function $F(t)$ at $t=t_0$

- $F_0 = 30 \text{N}$
- $k = 1000 \text{N/m}$
- $\zeta = 0.1$
- $\omega_n = 3.16$
- $t_0 = 2$
Example 3.9.1 Analytical versus numerical

Response to step input

clear all

%% Analytical solution (example 3.2.1)

Fo=30; k=1000; wn=3.16; zeta=0.1; to=0;
theta=atan(zeta/(1-zeta^2));
wd=wn*sqrt(1-zeta^2);
t=0:0.01:12;

Heaviside=stepfun(t,to); % define Heaviside Step function for 0<t<12

xt = (Fo/k - Fo/(k*sqrt(1-zeta^2)) * exp(-zeta*wn*(t-to)) *
      cos (wd*(t-to)- theta)) * Heaviside(t-to);

plot(t,xt); hold on

%% Numerical Solution

xo=[0; 0];
rs=[0 12];
[t,x]=ode45('f',ts,xo);
plot(t,x(:,1),'r'); hold off

function v=f(t,x)
 Fo=30; k=1000; wn=3.16; zeta=0.1; to=0; m=k/wn^2;
 v=[x(2); x(2).*-2*zeta*wn + x(1).*-wn^2 + Fo/m*stepfun(t,to)];

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Matlab Code

```matlab
x0=[0;0];
ts=[0 12];
[t,x]=ode45('funct',ts,x0);
plot(t,x(:,1))
```

```matlab
function v=funct(t,x)
F0=30;
k=1000;
wn=3.16;
z=0.1;
t0=2;
m=k/(wn^2);
v=[x(2); x(2).*-2*z*wn+x(1).*-wn^2+F0/m*stepfun(t,t0)];
```

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% problem 3.19
m=1000;
E=3.8e9;
A=0.03;
L=2;
k=E*A/L;
t0=0.2;
F0=100;
global F0 k m t0

% numerical solution
x0=[0;0];
ts=[0 0.5];
[t,x]=ode45('f_3_19',ts,x0);
plot(t,x(:,1))

function v=f_3_19(t,x)
global F0 k m t0
A=x(2);
F=((1-t./t0).*stepfun(t,0))-((1-t./t0).*stepfun(t,t0))*F0/m;
B=(-k/m)*x(1)+F;
v=[A; B];
Displacement (m) vs. Time (s)

Variations in correct direction

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3.9 Nonlinear Response Properties

Euler integration formula:

\[ x(t_{i+1}) = x(t_i) + F(x(t_i)) \Delta t + f(t_i) \Delta t \]

Nonlinear term

Analytical solutions not available so we must interrogate the numerical solution
Example 3.10 cubic spring subject to pulse input

\[ m\ddot{x}(t) + c\dot{x}(t) + kx(t) + k_1x^3(t) = 1500[\Phi(t-t_1) - \Phi(t-t_2)] \]

The state space form is:

\[ \dot{x}_1(t) = x_2(t) \]
\[ \dot{x}_2(t) = -2\zeta \omega_n x_2(t) - \omega_n^2 x_1(t) - \alpha x_1^3(t) + 15[\Phi(t-t_1) - \Phi(t-t_2)] \]
Red (solid) is nonlinear response. Blue (dashed) is linear response.
Is there any justification? Yes, hardening nonlinear spring.
The first part is due to IC.
Matlab Code

clear all

xo=[0.01; 1];
ts=[0 8];

[t,x]=ode45('f',ts,xo);
plot(t,x(:,1)); hold on % The response of nonlinear system

[t,x]=ode45('f1',ts,xo);
plot(t,x(:,1),'--'); hold off % The response of linear system

%---------------------------------------------
function v=f(t,x)
m=100; k=2000; c=20; wn=sqrt(k/m); zeta=c/2/sqrt(m*k); Fo=1500; alpha=3;
t1=1.5; t2=5;
v=[x(2); x(2).*-2*zeta*wn + x(1).*-wn^2 - x(1)^3.*alpha+
Fo/m*(stepfun(t,t1)-stepfun(t,t2))];

%---------------------------------------------
function v=f1(t,x)
m=100; k=2000; c=20; wn=sqrt(k/m); zeta=c/2/sqrt(m*k); Fo=1500; alpha=0;
t1=1; t2=5;
v=[x(2); x(2).*-2*zeta*wn + x(1).*-wn^2 - x(1)^3.*alpha+
Fo/m*(stepfun(t,t1)-stepfun(t,t2))];
What good is this ability?

- Investigate pulse width
- Investigate parameter changes
- Investigate effect of initial conditions
- Design and Prediction

Because there are not many closed form solutions, or magnitude expressions design must be done by numerical simulation