3 General forced response

• So far, all of the driving forces have been sine or cosine excitations
• In this chapter we examine the response to any form of excitation such as
  – Impulse
  – Sums of sines and cosines
  – Any integrable function
Linear Superposition allows us to break up complicated forces into sums of simpler forces, compute the response and add to get the total solution.

If $x_1, x_2$ are solutions of a linear homogeneous equation, then

$$x = a_1 x_1 + a_2 x_2 \text{ is also a solution.}$$

If $x_1$ is the particular sol of $\ddot{x} + \omega_n^2 x = f_1$
and $x_2$ the particular sol of $\ddot{x} + \omega_n^2 x = f_2$

$$\Rightarrow a x_1 + b x_2 \text{ solves } \ddot{x} + \omega_n^2 x = a f_1 + b f_2$$
3.1 Impulse Response Function

\[ F(t) = \begin{cases} 
0 & \text{for } t < \tau - \varepsilon \\
\frac{\hat{F}}{2\varepsilon} & \text{for } \tau - \varepsilon < t < \tau + \varepsilon \\
0 & \text{for } t > \tau + \varepsilon 
\end{cases} \]

\( \varepsilon \) is a small positive number

Figure 3.1
From sophomore dynamics The impulse imparted to an object is equal to the change in the object's momentum:

\[ F(t) \Delta t = \int_{\tau-\epsilon}^{\tau+\epsilon} F(t) \, dt \]

\[ I(\epsilon) = \int_{\tau-\epsilon}^{\tau+\epsilon} F(t) \, dt = \int_{-\infty}^{\infty} F(t) \, dt \, N \cdot s \]

\[ = \frac{F}{2\epsilon} 2\epsilon = \hat{F} \]
We use the properties of impulse to define the impulse function:

\[ F(t - \tau) = 0, \quad t \neq \tau \]

\[ \int_{-\infty}^{\infty} F(t - \tau) \, dt = \hat{F} \]

If \( \hat{F} = 1 \), this is the Dirac Delta \( \delta(t) \)
The effect of an impulse on a spring-mass-damper is related to its change in momentum.

\[ F \Delta t = \Delta mv \]

Thus the response to impulse with zero IC is equal to the free response with IC: \( x_0 = 0 \) and \( v_0 = \frac{F \Delta t}{m} \)
Recall that the free response to just non zero initial conditions is:

The solution of:

\[ m\ddot{x} + c\dot{x} + kx = 0 \quad x(0) = x_0 \quad \dot{x}(0) = v_0 \]

in underdamped case:

\[ x(t) = e^{-\zeta\omega_n t} \left[ \left( \frac{v_0 + \zeta\omega_n x_0}{\omega_d} \right) \sin \omega_d t + x_0 \cos \omega_d t \right] \]
So for an underdamped system the impulse response is \( x_0 = 0 \)

\[
x(t) = \frac{\hat{F}e^{-\zeta\omega_n t}}{m\omega_d} \sin \omega_d t \quad \text{(response to } \hat{F}) \quad (3.6)
\]

\[
x(t) = \hat{F}h(t), \quad \text{where } h(t) = \frac{e^{-\zeta\omega_n t}}{m\omega_d} \sin \omega_d t \quad (3.8)
\]

**Response to an impulse at \( t = 0 \), and zero initial conditions**

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The response to an impulse is thus defined in terms of the impulse response function, $h(t)$.

So, the response to $\delta(t)$ is given by $h(t)$.

$$h(t) = \frac{e^{-\zeta \omega_n t}}{m \omega_d} \sin \omega_d t \quad (3.8)$$

What is the response to a unit impulse applied at a time different from zero?

The response to $\delta(t-\tau)$ is $h(t-\tau)$.

This is given on the following slide.
For the case that the impulse occurs at $\tau$ note that the effects of non-zero initial conditions and other forcing terms must be superimposed on this solution (see Equation (3.9))

For example: If two pulses occur at two different times then their impulse responses will superimpose.
Consider the undamped impulse response

Setting \( \zeta = 0 \) in the equation (3.8)
Response to unit impulse applied at \( t = \tau \),
i.e. \( \delta(t - \tau) \) is:

\[
h(t - \tau) = \frac{1}{m \omega_n \sin \omega_n (t - \tau)}
\]
Example 3.1.1 Design a camera mount with a vibration constraint

Consider example 2.1.3 of the security camera again only this time with an impulsive load
Using the stiffness and mass parameters of Example 2.1.3, does the system stay within vibration limits if hit by a 1 kg bird traveling at 72 kmh?

The natural frequency of the camera system is

\[ \omega_n = \sqrt{\frac{k}{m_c}} = \sqrt{\frac{3Ebh^3}{m_c l^3}} = \sqrt{\frac{3(7.1 \times 10^{10} \text{ N/m})(0.02 \text{ m})(0.02 \text{ m})^3}{(3 \text{ kg})(0.55)^3}} = 261.3 \text{ rad/s} \]

From equations (3.7) and (3.8) with \( \zeta = 0 \), the impulsive response is:

\[ x(t) = \frac{F \Delta t}{m_c \omega_n} \sin \omega_n t = \frac{m_b v}{m_c \omega_n} \sin \omega_n t \]

The magnitude of the response due to the impulse is thus

\[ X = \left| \frac{m_b v}{m_c \omega_n} \right| \]
Next compute the momentum of the bird to complete the magnitude calculation:

\[ m_b v = 1 \text{ kg} \times 72 \frac{\text{km}}{\text{hour}} \times \frac{1000 \text{ m}}{\text{km}} \times \frac{\text{hour}}{3600 \text{ s}} = 20 \text{ kg m/s} \]

Next use this value in the expression for the maximum value:

\[ X = \frac{m_b v}{m_c \omega_n} = \frac{20 \text{ kg m/s}}{3 \text{ kg} \times 261.3 \text{ rad/s}} = 0.026 \text{ m} \]

This max value exceeds the camera tolerance.
Example 3.1.2: two impacts, zero initial conditions. (double hit)

\[ m = 1 \text{ kg}, \ c = 0.5 \text{ kg/s}, \ k = 4 \text{ N/m} \]

\[ \hat{F} = 2 \text{ N} \cdot \text{s} \quad \text{and} \quad F(t) = 2\delta(t) + \delta(t - \tau) \]

\[ \omega_n = 2, \ \zeta = 0.125 \]

\[ x_1(t) = \frac{2e^{-\zeta\omega_n t}}{m\omega_d} \sin \omega_d t = 1.008e^{-0.25t} \sin(1.984t), \ t > 0 \]

\[ x_2(t) = 0.504e^{-0.25(t-\tau)} \sin(1.984(t - \tau)), \ t > \tau \]

\[ x(t) = x_1 + x_2 \]

\[ = \begin{cases} 
1.008e^{-0.25(t)} \sin(1.984t) & 0 < t < \tau \\
1.008e^{-0.25t} \sin(1.984t) + 0.504e^{-0.25(t-\tau)} \sin(1.984(t - \tau)) & t > \tau 
\end{cases} \]
Example 3.1.2 two impacts and initial conditions

\[ \ddot{x} + 2\dot{x} + 4x = \delta(t) - \delta(t - 4), \quad x_0 = 1 \text{ mm}, \quad \dot{x}_0 = -1 \text{ mm/s} \]

Solve three simple problems and add the results.

Homogeneous solution \((\omega_n = 2 \text{ rad/s}, \quad \zeta = 0.5, \quad \omega_d = \sqrt{3} \text{ rad/s})\)

\[
x_h(t) = e^{-\zeta \omega_n t} \left[ \frac{v_0 + x_0 \zeta \omega_n}{\omega_d} \sin \omega_d t + x_0 \cos \omega_d t \right]
\]

\[
= e^{-t} \left[ \frac{-1 + 1}{\sqrt{3}} \sin \sqrt{3} t + \cos \sqrt{3} t \right] = e^{-t} \cos \sqrt{3} t
\]

Note, no need to redo constants of integration for impulse excitation (others, yes)
Computation of the response to first impulse:

Treat $\delta(t)$ as $x_0 = 0$ and $v_0 = 1$, $0 < t < 4$

$$x_I(t) = e^{-\zeta \omega_n t} \left[ \frac{v_0}{\omega_d} \sin \omega_d t \right] = \frac{1}{\sqrt{3}} e^{-t} \sin \sqrt{3}t$$

$0 < t < 4$
Total Response for $0 < t < 4$

$$x_1(t) = x_h(t) + x_I(t)$$

$$= e^{-t} \left( \cos \sqrt{3}t + \frac{1}{\sqrt{3}} \sin \sqrt{3}t \right),$$

for $0 \leq t < 4$
Next compute the response to the second impulse:

\[ x_2 = \frac{-1}{\sqrt{3}} e^{-t+4} \sin \sqrt{3}(t - 4), \quad t > 4 \]

\[ = -\frac{e^{-t+4}}{\sqrt{3}} \sin \sqrt{3}(t - 4) \quad H(t - 4) \]

Here the Heaviside step function is used to “turn on” the response to the impulse at \( t = 4 \) seconds.
To get the total response add the partial solutions:

\[ x(t) = e^{-t} \left( \frac{1}{\sqrt{3}} \sin \sqrt{3}t + \cos \sqrt{3}t \right) - \frac{e^{-(t+4)}}{\sqrt{3}} \sin \sqrt{3}(t - 4)H(t - 4) \]

frist impulse

initial condition

second impulse
3.2 Response to an Arbitrary Input

The response to general force, \( F(t) \), can be viewed as a series of impulses of magnitude \( F(t_i)\Delta t \).

Response at time \( t \) due to the \( i^{th} \) impulse zero IC:

\[
x_i(t) = [F(t_i)\Delta t] \cdot h(t-t_i) \quad \text{for } t > t_i
\]

If \( t = t_i \) (the \( i^{th} \) time interval):

\[
x(t_i) = \sum_{i=1}^{l} [F(t_i)\Delta t] h(t - t_i)
\]

\( \Delta t \to 0, t_i \to \tau \Rightarrow \)

\[
x(t) = \int_{0}^{t} F(\tau) h(t - \tau) d\tau \quad (3.12)
\]
Properties of convolution integrals: It is symmetric meaning:

Let $\alpha = t - \tau, t$ fixed so that $\tau = t - \alpha$
and $d\tau = -d\alpha$. Also $\tau : 0 \rightarrow t \Rightarrow \alpha : t \rightarrow 0$

$$x(t) = \int_{0}^{t} F(\tau)h(t - \tau)d\tau = \int_{0}^{t} F(t - \alpha)h(\alpha)(-d\alpha)$$

$$= \int_{0}^{t} F(t - \alpha)h(\alpha)d\alpha$$
The convolution integral, or Duhamel integral, for underdamped systems is:

\[
x(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_n t} \int_0^t \left[ F(\tau)e^{\zeta\omega_n \tau} \sin \omega_d (t - \tau) \right] d\tau
\]

\[
= \frac{1}{m\omega_d} \int_0^t F(t - \tau)e^{-\zeta\omega_n \tau} \sin \omega_d \tau d\tau \quad (3.13)
\]

• The response to any integrable force can be computed with either of these forms
• Which form to use depends on which is easiest to compute
Example 3.2.1: Step function input

To solve apply (3.13):

\[ m\ddot{x} + c\dot{x} + kx = \begin{cases} 0 & 0 < t < t_0 \\ F_0 & t_0 \leq t \end{cases} \]

\[ x_0 = 0, \quad v_0 = 0, \quad 0 < \zeta < 1 \]

Figure 3.6 Step function

\[ x(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_d t} \int_0^{t_0} (0)e^{\zeta\omega_d \tau} \sin \omega_d (t - \tau) d\tau + \frac{1}{m\omega_d} e^{-\zeta\omega_d t} \int_{t_0}^{t} F_0 e^{\zeta\omega_d \tau} \sin \omega_d (t - \tau) d\tau \]

\[ = \frac{F_0}{m\omega_d} e^{-\zeta\omega_d t} \int_{t_0}^{t} e^{\zeta\omega_d \tau} \sin \omega_d (t - \tau) d\tau \]
Integrating (use a table, code or calculator) yields the solution:

\[ x(t) = \frac{F_0}{k} - \frac{F_0}{k\sqrt{1-\zeta^2}} e^{-\zeta\omega_n(t-t_0)} \cos(\omega_d(t-t_0) - \theta), \quad t \geq t_0 \] (3.15)

\[ \theta = \tan^{-1} \frac{\zeta}{\sqrt{1-\zeta^2}} \] (3.16)

Fig 3.7
Example: undamped oscillator under IC and constant force

For an undamped system:

\[ h(t) = \frac{1}{m\omega_n} \sin \omega_n t \]

The homogeneous solution is

\[ x_h = \frac{v_0}{\omega_n} \sin \omega_n t + x_0 \cos \omega_n t, \quad t < t_1 \]

Good until the applied force acts at \( t_1 \), then:

\[ x_{1\rightarrow2} = \int_{0}^{t} F(\tau)h(t-\tau)d\tau, \quad t_1 < t < t_2 \]

\[ = \int_{0}^{t_1} F(\tau)h(t-\tau)d\tau + \int_{t_1}^{t} F(\tau)h(t-\tau)d\tau \]
Next compute the solution between $t_1$ and $t_2$

For $t_1 < t < t_2$

$$x_{1\rightarrow 2} = \int_{t_1}^{t} F_0 \frac{1}{m\omega_n} \sin \omega_n (t - \tau) d\tau$$

$$= \frac{F_0}{m\omega_n} \left[ \frac{(-1)(-1)}{\omega_n} \cos \omega_n (t - \tau) \right]_{t_1}^{t}$$

$$= \frac{F_0}{m\omega_n^2} [1 - \cos \omega_n (t - t_1)]$$
Now compute the solution for time greater than $t_2$

For $t > t_2$

\[
x_{2 \rightarrow} = \int_0^{t_1} F(\tau)h(t-\tau)d\tau + \int_{t_1}^{t_2} F(\tau)h(t-\tau)d\tau + \int_{t_2}^{t} F(\tau)h(t-\tau)d\tau
\]

\[
= \frac{F_0}{m\omega_n} \left\{ \frac{1}{\omega_n} \cos \omega_n(t-\tau) \right\}_{t_1}^{t_2}
\]

\[
= \frac{F_0}{m\omega_n^2} [\cos \omega_n(t-t_2) - \cos \omega_n(t-t_1)]
\]
Total solution is superposition:

\[
x(t) = \begin{cases} 
\frac{v_0}{\omega_n} \sin \omega_n t + x_0 \cos \omega_n t & t < t_1 \\
\frac{v_0}{\omega_n} \sin \omega_n t + x_0 \cos \omega_n t + \frac{F_0}{m \omega_n^2} [1 - \cos \omega_n (t - t_1)] & t_1 < t < t_2 \\
\frac{v_0}{\omega_n} \sin \omega_n t + x_0 \cos \omega_n t + \frac{F_0}{m \omega_n^2} [\cos \omega_n (t - t_2) - \cos \omega_n (t - t_1)] & t > t_2
\end{cases}
\]

\[
m = F_0 = 1, \omega_n = \sqrt{8}, t_1 = 2, t_2 = 4, x_0 = 0.1, v_0 = 0
\]

Check points: \( x \) increases after application of \( F \). Undamped response around \( x = 0 \)
Example 3.2.3: Static versus dynamic load

\[ m\ddot{x} + c\dot{x} + kx = \begin{cases} m_d g & t \geq 0 \\ 0 & t < 0 \end{cases} \]

This has max value of \( x_{\text{max}} = 2 \frac{m_d g}{k} \), twice the static load
Numerical simulation and plotting

• At the end of this chapter, numerical simulation is used to solve the problems of this section.
• Numerical simulation is often easier than computing these integrals.
• It is wise to check the two approaches against each other by plotting the analytical solution and numerical solution on the same graph.
3.3 Response to an Arbitrary Periodic Input

\[ \ddot{x} + 2\zeta \omega_n \dot{x} + \omega_n^2 x = F(t) \] where \( F(t) = F(t + T) \)

- We have solutions to sine and cosine inputs.
- What about periodic but non-harmonic inputs?
- We know that periodic functions can be represented by a series of sines and cosines (Fourier)
- Response is superposition of as many RHS terms as you think are necessary to represent the forcing function accurately

![Figure 3.11](image-url)
Recall the Fourier Series Definition:

Assume \( F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \Omega_n t + b_n \sin \Omega_n t \right) \) \hspace{1cm} (3.20)

where \( \Omega_n = \frac{2\pi n}{T} = n\omega \)

\( a_0 = \frac{2}{T} \int_{0}^{T} F(t) \, dt \) \hspace{1cm} (3.21) : twice the average

\( a_n = \frac{2}{T} \int_{0}^{T} F(t) \cos \Omega_n t \, dt \) \hspace{1cm} (3.22) : Oscillations around average

\( b_n = \frac{2}{T} \int_{0}^{T} F(t) \sin \Omega_n t \, dt \) \hspace{1cm} (3.23)
The terms of the Fourier series satisfy orthogonality conditions:

\[
\int_0^T \sin(n\omega_T t) \sin(m\omega_T t) dt = \begin{cases} 
0 & m \neq n \\
\frac{T}{2} & m = n
\end{cases} \tag{3.24}
\]

\[
\int_0^T \cos(n\omega_T t) \cos(m\omega_T t) dt = \begin{cases} 
0 & m \neq n \\
\frac{T}{2} & m = n
\end{cases} \tag{3.25}
\]

\[
\int_0^T \cos(n\omega_T t) \sin(m\omega_T t) dt = 0 \tag{3.26}
\]
Fourier Series Example

Step 1: find the F.S. and determine how many terms you need

\[ F(t) = \begin{cases} 
0, & t < t_1 \\
\frac{F_0}{t_2 - t_1}(t - t_1), & t_1 < t \leq t_2 
\end{cases} \]
Fourier Series Example

The graph illustrates the Fourier series approximation of a function $F(t)$ over time ($s$) and force ($F(t)$). The graph shows the function approximated with different numbers of coefficients: 2, 10, and 100. Each curve represents the accuracy of the approximation, with the more coefficients leading to a closer match to the original function.

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Having obtained the FS of input

• The next step is to find responses to each term of the FS
• And then, just add them up!
• Danger!!: Resonance occurs whenever a multiple of excitation frequency equals the natural frequency.
• You may excite at 100rad/s and observe resonance while natural frequency is 500rad/s!!
Solution as a series of sines and cosines to
\[
\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = F(t)
\]
The solution can be written as a summation
\[
x_p(t) = x_0(t) + \sum_{n=1}^{\infty} x_{cn}(t) + x_{sn}(t)
\]
where \(x_0(t)\) is a solution to
\[
\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = \frac{a_0}{2} \Rightarrow x_0(t) = \frac{a_0}{2\omega_n^2}
\]
and \(x_{cn}(t)\) and \(x_{sn}(t)\) are solutions to
\[
\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = a_n \cos(n\omega_T t)
\]
\[
\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = b_n \sin(n\omega_T t)
\]

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3.4 Transform Methods

An alternative to solving the previous problems, similar to section 2.3
Laplace Transform

• Laplace transformation

\[ F(s) = \int_0^\infty f(t) e^{-st} \, dt = \mathcal{L}\{f(t)\} \quad (3.41) \]

Laplace transforms are very useful because they change differential equations into simple algebraic equations.

• Examples of Laplace transforms (see page 216 in book)

<table>
<thead>
<tr>
<th>( f(t) )</th>
<th>( F(s) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Step function, ( u(t) )</td>
<td>( 1/s )</td>
</tr>
<tr>
<td>( e^{-at} )</td>
<td>( 1/(s+a) )</td>
</tr>
<tr>
<td>( \sin(\omega t) )</td>
<td>( \omega / (s^2 + \omega^2) )</td>
</tr>
</tbody>
</table>
Laplace Transform

• Example: Laplace transform of a step function $u(t)$

$$L\{u(t)\} = \int_{0}^{\infty} e^{-st} dt = \left[ \frac{-e^{-st}}{s} \right]_{0}^{\infty} = \frac{1}{s}$$

• Example: Laplace transform of $e^{-at}$

$$L\{e^{-at}\} = \int_{0}^{\infty} e^{-at} e^{-st} dt = \int_{0}^{\infty} e^{-(s+a)t} dt$$

$$L\{e^{-at}\} = \left[ \frac{-e^{-(s+a)t}}{s + a} \right]_{0}^{\infty} = \frac{1}{s + a}$$
Laplace Transforms of Derivatives

• Laplace transform of the derivative of a function

\[
L \left\{ \frac{df(t)}{dt} \right\} = \int_{0}^{\infty} \frac{df(t)}{dt} e^{-st} \, dt
\]

Integration by parts gives,

\[
L \left\{ \frac{df(t)}{dt} \right\} = \left[ f(t)e^{-st} \right]_{0}^{\infty} + s\int_{0}^{\infty} f(t)e^{-st} \, dt
\]

\[
L \left\{ \frac{df(t)}{dt} \right\} = -f(0) + sL \{ f(t) \}
\]
Laplace Transform Procedures

• Laplace transform of the integral of a function

\[ L\left\{ \int_{-\infty}^{t} f(t)dt \right\} = \frac{1}{s} L\{f(t)\} + \int_{-\infty}^{0} f(t)dt \]

Steps in using the Laplace transformation to solve DE’s

• Find differential equations
• Find Laplace transform of equations
• Rearrange equations in terms of variable of interest
• Convert back into time domain to find resulting response (inverse transform using tables)
Laplace Transform Shift Property

Note these shift properties in $t$ and $s$ spaces...

\[
e^{at} f(t) \xrightarrow{L} F(s - a)
\]

\[
f(t - a) \Phi(t - a) \xrightarrow{L} e^{-as} F(s)
\]

thus

\[
\delta(t) \xrightarrow{L} 1 \quad \Rightarrow \quad \delta(t - a) \xrightarrow{L} e^{-as}
\]
Example 3.4.3: compute the forced response of a spring mass system to a step input using LT

The equation of motion is

\[ m\ddot{x}(t) + kx(t) = \Phi(t) \]

Taking the Laplace Transform (zero initial conditions)

\[ (ms^2 + k)X(s) = \frac{1}{s} \Rightarrow X(s) = \frac{1}{s(ms^2 + k)} = \frac{1/m}{s(s^2 + \omega_n^2)} \]

Taking the inverse Laplace Transform yields:

\[ x(t) = \frac{1/m}{\omega_n^2} \left(1 - \cos \omega_n t\right) = \frac{1}{k} \left(1 - \cos \omega_n t\right) \]

Compare this to the solution given in (3.18)
### Fourier Transform

- From Fourier series of non-periodic functions
- Allow period to go to infinity
- Similar to Laplace Transform
- Useful for random inputs

\[
X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, dt
\]

- Corresponding inverse transform

\[
x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} \, d\omega
\]

- Fourier transform of the unit impulse response is the frequency response function

\[
H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} \, dt
\]