2.4 Base Excitation

- Important class of vibration analysis
  - Preventing excitations from passing from a vibrating base through its mount into a structure

- Vibration isolation
  - Vibrations in your car
  - Satellite operation
  - Disk drives, etc.
FBD of SDOF Base Excitation

\[ F = -k(x - y) - c(\dot{x} - \dot{y}) = m\ddot{x} \]

\[ m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky \quad (2.61) \]

System Sketch

System FBD
SDOF Base Excitation (cont)

Assume: \( y(t) = Y \sin(\omega t) \) and plug into Equation (2.61)

\[
m\ddot{x} + c\dot{x} + kx = c\omega Y \cos(\omega t) + kY \sin(\omega t) \quad (2.63)
\]

harmonic forcing functions

For a car, \( \omega = \frac{2\pi}{\tau} = \frac{2\pi V}{\lambda} \)

The steady-state solution is just the superposition of the two individual particular solutions (system is linear).

\[
\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = 2\zeta\omega_n \omega Y \cos(\omega t) + \omega_n^2 Y \sin(\omega t) \quad (2.64)
\]
Particular Solution (sine term)

With a sine for the forcing function,

\[ \ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = f_0 \sin \omega t \]

\[ x_{ps} = A_s \cos \omega t + B_s \sin \omega t = X_s \sin(\omega t - \phi_s) \]

where

\[ A_s = \frac{-2\zeta\omega_n \omega f_0}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n \omega)^2} \]

\[ B_s = \frac{(\omega_n^2 - \omega^2) f_0}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n \omega)^2} \]

Use rectangular form to make it easier to add the cos term

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Particular Solution (cos term)

With a cosine for the forcing function, we showed

\[ \ddot{x} + 2\zeta \omega_n \dot{x} + \omega_n^2 x = f_{0c} \cos \omega t \]

\[ x_{pc} = A_c \cos \omega t + B_c \sin \omega t = X_c \cos(\omega t - \phi_c) \]

where

\[ A_c = \frac{(-\omega_n^2 + \omega^2)f_{0c}}{(\omega_n^2 - \omega^2)^2 + (2\zeta \omega_n \omega)^2} \]

\[ B_c = \frac{2\zeta \omega_n \omega f_{0c}}{(\omega_n^2 - \omega^2)^2 + (2\zeta \omega_n \omega)^2} \]
Magnitude $X/Y$

Now add the sin and cos terms to get the magnitude of the full particular solution

$$X = \sqrt{\frac{f_{0c}^2 + f_{0s}^2}{(\omega_n^2 - \omega^2)^2 + (2\zeta \omega_n \omega)^2}} = \omega_n Y \sqrt{\frac{(2\zeta \omega)^2 + \omega_n^2}{(\omega_n^2 - \omega^2)^2 + (2\zeta \omega_n \omega)^2}}$$

where $f_{0c} = 2\zeta \omega_n \omega Y$ and $f_{0s} = \omega_n^2 Y$

if we define $r = \frac{\omega}{\omega_n}$ this becomes

$$X = Y \sqrt{\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}} \quad (2.70)$$

$$\frac{X}{Y} = \sqrt{\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}} \quad (2.71)$$
The relative magnitude plot of \( \frac{X}{Y} \) versus frequency ratio: Called the \textit{Displacement Transmissibility}.

Figure 2.13
From the plot of relative Displacement Transmissibility observe that:

- \( X/Y \) is called Displacement Transmissibility Ratio
- Potentially severe amplification at resonance
- Attenuation for \( r > \sqrt{2} \) Isolation Zone
- If \( r < \sqrt{2} \) transmissibility decreases with damping ratio Amplification Zone
- If \( r >> 1 \) then transmissibility increases with damping ratio \( X_p \sim 2Y\zeta/r \)
Next examine the **Force Transmitted to the mass** as a function of the frequency ratio

\[ F_T = -k(x - y) - c(\dot{x} - \dot{y}) = m\ddot{x} \]

At steady state, \( x(t) = X \cos(\omega t - \phi) \), so \( \ddot{x} = -\omega^2 X \cos(\omega t - \phi) \)

\[ |F_T| = m\omega^2 X = k r^2 X \]
Plot of Force Transmissibility (in dB) versus frequency ratio

Figure 2.14
Figure 2.15 Comparison between force and displacement transmissibility

![Graph comparing force and displacement transmissibility](image-url)
Example 2.4.1: Effect of speed on the amplitude of car vibration

$m = \text{mass of car}$

$\dot{x}(t) \quad \text{Velocity of car}$

Suspension system

Neglected unsprung mass

Road surface

$0.02 \, \text{m}$

$6 \, \text{m}$
Model the road as a sinusoidal input to base motion of the car model

Approximation of road surface:
\[ y(t) = (0.01 \text{ m}) \sin \omega_b t \]

\[ \omega_b = v(\text{km/hr}) \left( \frac{1}{0.006 \text{ km}} \right) \left( \frac{\text{hour}}{3600 \text{ s}} \right) \left( \frac{2\pi \text{ rad}}{\text{cycle}} \right) = 0.2909v \text{ rad/s} \]

\[ \omega_b (20\text{km/hr}) = 5.818 \text{ rad/s} \]

From the data give, determine the frequency and damping ratio of the car suspension:

\[ \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4 \times 10^4 \text{ N/m}}{1007 \text{ kg}}} = 6.303 \text{ rad/s} \ (\approx 1 \text{ Hz}) \]

\[ \zeta = \frac{c}{2\sqrt{km}} = \frac{2000 \text{ Ns/m}}{2\sqrt{(4 \times 10^4 \text{ N/m})(1007 \text{ kg})}} = 0.158 \]
From the input frequency, input amplitude, natural frequency and damping ratio use equation (2.70) to compute the amplitude of the response:

\[ r = \frac{\omega_b}{\omega} = \frac{5.818}{6.303} \]

\[ X = Y \sqrt{\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}} \]

\[ = (0.01 \, \text{m}) \sqrt{\frac{1 + \left[2(0.158)(0.923)\right]^2}{\left(1 - (0.923)^2\right)^2 + \left(2(0.158)(0.923)\right)^2}} = 0.0319 \, \text{m} \]

What happens as the car goes faster? See Table 2.1.
Example 2.4.2: Compute the force transmitted to a machine through base motion at resonance

From (2.77) at $r = 1$:

$$\frac{F_T}{kY} = \left[\frac{1 + (2\zeta)^2}{(2\zeta)^2}\right]^{1/2} \Rightarrow F_T = \frac{kY}{2\zeta} \sqrt{1 + 4\zeta^2}$$

From given $m$, $c$, and $k$: $\zeta = \frac{c}{2\sqrt{km}} = \frac{900}{2\sqrt{40,000\times3000}} \approx 0.04$

From measured excitation $Y = 0.001$ m:

$$F_T = \frac{kY}{2\zeta} \sqrt{1 + 4\zeta^2} = \frac{(40,000 \text{ N/m})(0.001 \text{ m})}{2(0.04)} \sqrt{1 + 4(0.04)^2} = 501.6 \text{ N}$$
2.5 Rotating Unbalance

- Gyros
- Cryo-coolers
- Tires
- Washing machines

$e = \text{eccentricity}$

$m_0 = \text{mass unbalance}$

$\omega = \text{rotation frequency}$
Rotating Unbalance (cont)

What force is imparted on the structure? Note it rotates with x component:

\[ x_r = e \sin \omega_r t \]

\[ \Rightarrow a_x = \ddot{x}_r = -e \omega_r^2 \sin \omega_r t \]

From sophomore dynamics,

\[ R_x = m_0 a_x = -m_o e \omega_r^2 \sin \theta = -m_o e \omega_r^2 \sin \omega_r t \]

\[ R_y = m_0 a_y = -m_o e \omega_r^2 \cos \theta = -m_o e \omega_r^2 \cos \omega_r t \]
Rotating Unbalance (cont)

The problem is now just like any other SDOF system with a harmonic excitation

\[ m \ddot{x} + c \dot{x} + kx = m_0 e^{\omega_r^2} \sin \omega_r t \quad (2.82) \]

or

\[ \ddot{x} + 2\zeta \omega_n \dot{x} + \omega_n^2 x = \frac{m_0}{m} e^{\omega_r^2} \sin \omega_r t \]

Note the influences on the forcing function (we are assuming that the mass \( m \) is held in place in the \( y \) direction as indicated in Figure 2.18)
Rotating Unbalance (cont)

- Just another SDOF oscillator with a harmonic forcing function
- Expressed in terms of frequency ratio $r$

\[
x_p(t) = X \sin(\omega_r t - \phi) \quad (2.83)
\]

\[
X = \frac{m_o e}{m} \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \quad (2.84)
\]

\[
\phi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) \quad (2.85)
\]
Figure 2.20: Displacement magnitude vs frequency caused by rotating unbalance
Example 2.5.1: Given the deflection at resonance (0.1 m), $\zeta = 0.05$ and a 10% out of balance, compute $e$ and the amount of added mass needed to reduce the maximum amplitude to 0.01 m.

At resonance $r = 1$ and

$$\frac{mX}{m_0 e} = \frac{1}{2\zeta} = \frac{1}{2(0.05)} \Rightarrow 10 \frac{0.1 \text{ m}}{e} = \frac{1}{2\zeta} = 10 \Rightarrow e = 0.1 \text{ m}$$

Now to compute the added mass, again at resonance;

$$\frac{m}{m_0} \left( \frac{X}{0.1 \text{ m}} \right) = 10 \quad \text{Use this to find } \Delta m \text{ so that } X \text{ is 0.01:}$$

$$\frac{m + \Delta m}{m_0} \left( \frac{0.01 \text{ m}}{0.1 \text{ m}} \right) = 10 \Rightarrow \frac{m + \Delta m}{(0.1)m} = 100 \Rightarrow \Delta m = 9m$$

Here $m_0$ is 10%$m$ or 0.1$m$
Example 2.5.2 Helicopter rotor unbalance

Given

\[ k = 1 \times 10^5 \text{ N/m} \]
\[ m_{\text{tail}} = 60 \text{ kg} \]
\[ m_{\text{rot}} = 20 \text{ kg} \]
\[ m_0 = 0.5 \text{ kg} \]
\[ \zeta = 0.01 \]

Compute the deflection at 1500 rpm and find the rotor speed at which the deflection is maximum.
Example 2.5.2 Solution

The rotating mass is 20 + 0.5 or 20.5. The stiffness is provided by the Tail section and the corresponding mass is that determined in Example 1.4.4. So the system natural frequency is

\[ \omega_n = \sqrt{\frac{k}{m + \frac{m_{\text{tail}}}{3}}} = \sqrt{\frac{10^5 \text{ N/m}}{20.5 + \frac{60 \text{ kg}}{3}}} = 46.69 \text{ rad/s} \]

The frequency of rotation is

\[ \omega_r = 1500 \text{ rpm} = 1500 \frac{\text{rev}}{\text{min}} \frac{\text{min}}{60 \text{ s}} \frac{2\pi \text{ rad}}{\text{rev}} = 157 \text{ rad/s} \]

\[ \Rightarrow r = \frac{157 \text{ rad/s}}{49.49 \text{ rad/s}} = 3.16 \]
Now compute the deflection at \( r = 3.16 \) and \( \zeta = 0.01 \) using eq (2.84)

\[
X = \frac{m_0e}{m} \frac{r^2}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}
\]

\[
= \frac{(0.5 \text{ kg})(0.15 \text{ m})}{20.5 \text{ kg}} \frac{(3.16)^2}{\sqrt{(1 - (3.16)^2)^2 - (2(0.01)(3.16))^2}} = 0.004 \text{ m}
\]

At around \( r = 1 \), the max deflection occurs:

\[
r = 1 \Rightarrow \omega_r = 49.69 \text{ rad/s} = 49.69 \frac{\text{rad}}{\text{s}} \frac{\text{rev}}{2\pi \text{ rad}} \frac{60 \text{ s}}{\text{min}} = 474.5 \text{ rpm}
\]

At \( r = 1 \):

\[
X = \frac{(0.5 \text{ kg})(0.15 \text{ m})}{20.5 \text{ kg}} \frac{1}{2(0.01)} = 0.183 \text{ m} \text{ or } 18.3 \text{ cm}
\]
2.6 Measurement Devices

- A basic transducer used in vibration measurement is the accelerometer.
- This device can be modeled using the base equations developed in the previous section:

\[ \sum F = -k(x-y) - c(\dot{x} - \dot{y}) = m\ddot{x} \]

\[ \Rightarrow m\ddot{x} = -c(\dot{x} - \dot{y}) - k(x - y) \]

(2.86) and (2.61)

Here, \( y(t) \) is the measured response of the structure.
Base motion applied to measurement devices

Let \( z(t) = x(t) - y(t) \) (2.87):

\[
\Rightarrow \quad m \ddot{z} + c \dot{z}(t) + k z(t) = m \omega_b^2 Y \cos \omega_b t \quad (2.88)
\]

\[
\Rightarrow \quad \frac{Z}{Y} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \quad (2.90)
\]

and

\[
\theta = \tan^{-1}\left( \frac{2\zeta r}{1-r^2} \right) \quad (2.91)
\]

These equations should be familiar from base motion. Here they describe measurement!
Magnitude and sensitivity plots for accelerometers.

Fig 2.26
Magnitude plot showing Regions of measurement
In the accel region, output voltage is nearly proportional to displacement

Fig 2.27
Effect of damping on proportionality constant

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2.7 Other forms of damping

<table>
<thead>
<tr>
<th>Name</th>
<th>Damping Force</th>
<th>$c_{eq}$</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear viscous damping</td>
<td>$c\dot{x}$</td>
<td>$c$</td>
<td>Slow fluid</td>
</tr>
<tr>
<td>Air damping</td>
<td>$a \text{sgn}(\dot{x})x^2$</td>
<td>$\frac{8a\omega X}{3\pi}$</td>
<td>Fast fluid</td>
</tr>
<tr>
<td>Coulomb damping</td>
<td>$\beta \text{sgn} \dot{x}$</td>
<td>$\frac{4\beta}{\pi\omega X}$</td>
<td>Sliding friction</td>
</tr>
<tr>
<td>Displacement-squared damping</td>
<td>$d \text{sgn}(\dot{x})x^2$</td>
<td>$\frac{4dX}{3\pi\omega}$</td>
<td>Material damping</td>
</tr>
<tr>
<td>Solid, or structural, damping</td>
<td>$b \text{sgn}(\dot{x})</td>
<td>x</td>
<td>$</td>
</tr>
</tbody>
</table>

These various other forms of damping are all nonlinear. They can be compared to linear damping by the method of “equivalent viscous damping” discussed next. A numerical treatment of the exact response is given in section 2.9.
The method of equivalent viscous damping: consists of comparing the energy dissipated during one cycle of forced response.

Assume a steady state resulting from a harmonic input and compute the energy dissipated per one cycle:

\[ x_{ss} = X \sin \omega t \]

The energy per cycle for a viscously damped system is

\[ \Delta E = \int F_x dx = \int_0^{2\pi/\omega} c \dot{x} \frac{dx}{dt} dt = \int_0^{2\pi/\omega} c \dot{x}^2 dt \quad (2.99) \]

\[ x_{ss} = X \sin \omega t \Rightarrow \dot{x} = \omega X \cos \omega t \Rightarrow \]

\[ \Delta E = c \int_0^{2\pi/\omega} \left( \omega X \cos \omega t \right)^2 dt = \pi c \omega X^2 \quad (2.101) \]

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Next compute the energy dissipated per cycle for Coulomb damping:

\[
\Delta E = \mu mg \int_{0}^{2\pi/\omega} \text{sgn}(\dot{x})\dot{x} dt = \mu mg
\]

\[
= \mu mgX \left( \int_{0}^{\pi/2} \cos u du - \int_{\pi/2}^{3\pi/2} \cos u du + \int_{3\pi/2}^{2\pi} \cos u du \right) = 4\mu mgX
\]

Here we let \( u = \omega t \) and \( du = \omega dt \) and split up the integral according to the sign changes in velocity. Next compare this energy to that of a viscous system:

\[
\pi c_{eq} \omega X^2 = 4\mu mgX \Rightarrow c_{eq} = \frac{4\mu mg}{\pi \omega X} \quad (2.105)
\]

This yields a linear viscous system dissipating the same amount of energy per cycle.
Using the equivalent viscous damping calculations, each of the systems in Table 2.2 can be approximated by a linear viscous system. In particular, $c_{eq}$ can be used to derive amplitude expressions. However, as indicated in Section 2.8 and 2.9 the response can be simulated numerically to provide more accurate magnitude and response information.
Hysteresis: an important concept characterizing damping

- A plot of displacement versus spring/damping force for viscous damping yields a loop.
- At the bottom is a stress-strain plot for a system with material damping of the hysteretic type.
- The enclosed area is equal to the energy lost per cycle.
The measured area yields the energy dissipated. For some materials, called hysteretic this is

\[ \Delta E = \pi k \beta X^2 \quad (2.120) \]

Here the constant \( \beta \), a measured quantity is called the hysteretic damping constant, \( k \) is the stiffness and \( X \) is the amplitude.

Comparing this to the viscous energy yields:

\[ c_{eq} = \frac{k \beta}{\omega} \]
Hysteresis gives rise to the concept of complex stiffness.

Substitution of the equivalent damping coefficient and using the complex exponential to describe a harmonic input yields:

\[ m\ddot{x} + \frac{k\beta}{\omega} \dot{x} + \omega_n^2 x = F_0 e^{j\omega t} \]

Assuming \( x(t) = Xe^{j\omega t} \) and \( \dot{x}(t) = Xj\omega e^{j\omega t} \)

yields

\[ m\ddot{x}(t) + k(1 + j\beta) x(t) = F_0 e^{j\omega t} \]

complex stiffness
2.8 Numerical Simulation and Design

• Four things we can do computationally to help solve, understand and design vibration problems subject to harmonic excitation
• Symbolic manipulation
• Plotting of the time response
• Solution and plotting of the time response
• Plotting magnitude and phase
Symbolic Manipulation

Let

\[ A = \begin{bmatrix} \omega_n^2 - \omega^2 & 2\zeta \omega_n \omega \\ -2\zeta \omega_n \omega & \omega_n^2 - \omega^2 \end{bmatrix} \]

and

\[ x = \begin{bmatrix} f_0 \\ 0 \end{bmatrix} \]

What is

\[ A_n = A^{-1}x \]

This can be solved using Matlab, Mathcad or Mathematica
Symbolic Manipulation

Solve equations (2.34) using Mathcad symbolics:

Enter this $\begin{bmatrix} \omega_n^2 - \omega^2 & 2\cdot\zeta\cdot\omega_n\cdot\omega \\ -\left(2\cdot\zeta\cdot\omega_n\cdot\omega\right) & \omega_n^2 - \omega^2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} f_0 \\ 0 \end{bmatrix}$

Choose evaluate under symbolics to get this

\[
\begin{bmatrix}
\frac{\omega_n^4 - 2\cdot\omega_n^2\cdot\omega + \omega^4 + 4\cdot\zeta^2\cdot\omega_n^2\cdot\omega}{\omega_n^4 - 2\cdot\omega_n^2\cdot\omega + \omega^4 + 4\cdot\zeta^2\cdot\omega_n^2\cdot\omega} \cdot f_0 \\
2\cdot\zeta\cdot\omega_n\cdot\omega \cdot \frac{\omega_n^4 - 2\cdot\omega_n^2\cdot\omega + \omega^4 + 4\cdot\zeta^2\cdot\omega_n^2\cdot\omega}{\omega_n^4 - 2\cdot\omega_n^2\cdot\omega + \omega^4 + 4\cdot\zeta^2\cdot\omega_n^2\cdot\omega} \cdot f_0
\end{bmatrix}
\]
In MATLAB Command Window

```
>> syms z wn w f0
>> A=[wn^2-w^2 2*z*wn*w;-2*z*wn*w wn^2-w^2];
>> x=[f0 0];
>> An=inv(A)*x
   An =
     [ (wn^2-w^2)/(wn^4-2*wn^2*w^2+w^4+4*z^2*wn^2*w^2)*f0]
     [   2*z*wn*w/(wn^4-2*wn^2*w^2+w^4+4*z^2*wn^2*w^2)*f0]
>> pretty(An)
```

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%m-file to plot base excitation to mass vibration
r=linspace(0,3,500);
ze=[0.01;0.05;0.1;0.2;0.5];
X=sqrt( ((2*ze*r).^2+1) ./ ( (ones(size(ze))*(1-r.*r).^2) + (2*ze*r).^2) );
figure(1)
plot(r,20*log10(X))

The values of $\zeta$ can then be chosen directly off of the plot.

For Example:
If the T.R. needs to be less than 2 (or 6dB) and $r$ is close to 1 then $\zeta$ must be more than 0.2 (probably about 0.3).
%m-file to plot base excitation to mass vibration
r=linspace(0,3,500);
ze=[0.01;0.05;0.1;0.20;0.50];
X=sqrt( ((2*ze*r).^2+1) ./ ( (ones(size(ze))*(1-r.*r).^2) + (2*ze*r).^2) );
F=X.*(ones(length(ze),1)*r).^2;
figure(1)
plot(r,20*log10(F))
Numerical Simulation

We can put the forced case:

\[ m \ddot{x}(t) + c \dot{x}(t) + k x(t) = F_0 \cos \omega t \]

\[ \ddot{x}(t) + 2 \zeta \omega_n \dot{x}(t) + \omega_n^2 x(t) = f_0 \cos \omega t \]

Into a state space form

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -2 \zeta \omega_n x_2 - \omega_n^2 x_1 + f_0 \cos \omega t \\
\dot{x}(t) &= Ax(t) + f(t), \quad f(t) = \begin{bmatrix} 0 \\ f_0 \cos \omega t \end{bmatrix}
\end{align*}
\]
Numerical Integration

Euler: \( x(t_{i+1}) = x(t_i) + Ax(t_i)\Delta t + f(t_i)\Delta t \)

Using the ODE45 function

\[
\begin{align*}
&\text{>>TSPAN=[0 10];} \\
&\text{>>Y0=[0;0];} \\
&\text{>>[t,y] =ode45('num_for',TSPAN,Y0);} \\
&\text{>>plot(t,y(:,1))}
\end{align*}
\]

Including forcing

\[
\begin{verbatim}
function Xdot=num_for(t,X) m=100;k=1000;c=25; ze=c/(2*sqrt(k*m)); wn=sqrt(k/m); w=2.5;F=1000;f=F/m; f=[0 ;f*cos(w*t)]; A=[0 1;-wn*wn -2*ze*wn]; Xdot=A*X+f;
\end{verbatim}
\]
Example 2.8.2: Design damping for an electronics model

• 100 kg mass, subject to $150\cos(5t)$ N
• Stiffness $k=500$ N/m, $c = 10$ kg/s
• Usually $x_0=0.01$ m, $v_0 = 0.5$ m/s
• Find a new $c$ such that the max transient value is 0.2 m.
Response of the board is;

transient exceeds design specification value

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To run this use the following file:

```
function Xdot=num_for(t,X)
    m=100;k=500;c=10;
    ze=c/(2*sqrt(k*m));
    wn=sqrt(k/m);
    w=5;F=150;f=F/m;
    f=[0 ;f*cos(w*t)];
    A=[0 1;-wn*wn -2*ze*wn];
    Xdot=A*X+f;
```

Create function to model forcing

Matlab command window

```
>> TSPAN=[0 40];
>> Y0=[0.01;0.5];
>> [t,y] = ode45('num_for',TSPAN,Y0);
>> plot(t,y(:,1))
>> xlabel('Time (sec)')
>> ylabel('Displacement (m)')
>> grid
```

Rerun this code, increasing c each time until a response that satisfies the design limits results.
Solution: code it, plot it and change $c$ until the desired response bound is obtained.

Meets amplitude limit when $c=195 \text{kg/s}$
2.9 Nonlinear Response Properties

- More than one equilibrium
- Steady state depends on initial conditions
- Period depends on I.C. and amplitude
- Sub and super harmonic resonance
- No superposition
- Harmonic input resulting in nonperiodic motion
- Jumps appear in response amplitude
Computing the forced response of a non-linear system

A non-linear system has an equation of motion given by:

\[
\ddot{x}(t) + f(x, \dot{x}) = f_0 \cos \omega t
\]

Put this expression into state-space form:

\[
\begin{cases}
\dot{x}_1(t) = x_2(t) \\
\dot{x}_2(t) = -f(x_1, x_2) + f_0 \cos \omega t
\end{cases}
\]

In vector form:

\[
\dot{x}(t) = F(x) + f(t)
\]
Numerical form

Vector of nonlinear dynamics

\[ F(x) = \begin{bmatrix} x_2(t) \\ -f(x_1, x_2) \end{bmatrix}, \quad f(t) = \begin{bmatrix} 0 \\ f_0 \cos \omega t \end{bmatrix} \]

Euler equation is

\[ x(t_{i+1}) = x(t_i) + F(x(t_i))\Delta t + f(t_i)\Delta t \]
Cubic nonlinear spring (2.9.1)

\[ \ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x - \beta x^3 = f_0 \cos \omega t \]

Superharmonic resonance

\[ \omega = \frac{\omega_n}{2.964} \]

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Cubic nonlinear spring near resonance

\[ \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x - \beta x^3 = f_0 \cos \omega t \]

Response near linear resonance

\[ \omega = \frac{\omega_n}{1.09} \]

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