Chapter 2 Response to Harmonic Excitation

Introduces the important concept of resonance
2.1 Harmonic Excitation of Undamped Systems

- Consider the usual spring mass damper system with applied force $F(t) = F_0 \cos \omega t$
- $\omega$ is the driving frequency
- $F_0$ is the magnitude of the applied force
- We take $c = 0$ to start with
Equations of motion

- Solution is the sum of *homogenous* and *particular* solution
- The *particular* solution assumes form of forcing function (physically the input wins):

\[
m\ddot{x}(t) = -kx(t) + F_0 \cos(\omega t)
\]

\[
\ddot{x}(t) + \omega_n^2 x(t) = f_0 \cos(\omega t)
\]

where \( f_0 = \frac{F_0}{m} \), \( \omega_n = \sqrt{ \frac{k}{m} } \)
Substitute *particular* solution into the equation of motion:

\[
\begin{align*}
\ddot{x}_p &= \omega_n^2 x_p \\
-\omega^2 X \cos \omega t + \omega_n^2 X \cos \omega t &= f_0 \cos \omega t \\
\end{align*}
\]

solving yields: \( X = \frac{f_0}{\omega_n^2 - \omega^2} \)

Thus the particular solution has the form:

\[
x_p(t) = \frac{f_0}{\omega_n^2 - \omega^2} \cos(\omega t)
\]
Add particular and homogeneous solutions to get general solution:

\[ x(t) = \left( A_1 \sin \omega_n t + A_2 \cos \omega_n t \right) + \frac{f_0}{\omega_n^2 - \omega^2} \cos \omega t \]

\( A_1 \) and \( A_2 \) are constants of integration.
Apply the initial conditions to evaluate the constants

\[ x(0) = A_1 \sin 0 + A_2 \cos 0 + \frac{f_0}{\omega_n^2 - \omega^2} \cos 0 = A_2 + \frac{f_0}{\omega_n^2 - \omega^2} = x_0 \]

\[ \Rightarrow A_2 = x_0 - \frac{f_0}{\omega_n^2 - \omega^2} \]

\[ \dot{x}(0) = \omega_n (A_1 \cos 0 - A_2 \sin 0) - \frac{f_0}{\omega_n^2 - \omega^2} \sin 0 = \omega_n A_1 = v_0 \]

\[ \Rightarrow A_1 = \frac{v_0}{\omega_n} \]

\[ x(t) = \frac{v_0}{\omega_n} \sin \omega_n t + \left( x_0 - \frac{f_0}{\omega_n^2 - \omega^2} \right) \cos \omega_n t + \frac{f_0}{\omega_n^2 - \omega^2} \cos \omega t \quad (2.11) \]
Comparison of free and forced response

- Sum of two harmonic terms of different frequency
- Free response has amplitude and phase effected by forcing function
- Our solution is not defined for $\omega_n = \omega$ because it produces division by 0.
- If forcing frequency is close to natural frequency the amplitude of particular solution is very large
Response for \( m=100 \) kg, \( k=1000 \) N/m, \( F=100 \) N, \( \omega = \omega_n + 5 \) \( v_0=0.1 \) m/s and \( x_0= -0.02 \) m.

Note the obvious presence of two harmonic signals
What happens when $\omega$ is near $\omega_n$?

When the drive frequency and natural frequency are close a **beating** phenomena occurs.

$$x(t) = \frac{2f_0}{\omega_n^2 - \omega^2} \sin\left(\frac{\omega_n - \omega}{2} t\right) \sin\left(\frac{\omega_n + \omega}{2} t\right)$$  \hspace{1em} (2.13)
What happens when $\omega$ is $\omega_n$?

$x_p(t) = tX \sin(\omega t)$

substitute into eq. and solve for $X$

$$X = \frac{f_0}{2\omega}$$

When the drive frequency and natural frequency are the same the amplitude of the vibration grows without bounds. This is known as a **resonance** condition. The most important concept in Chapter 2!
Example 2.1.1: Compute and plot the response for 
m=10 kg, \(k=1000 \text{ N/m}, \ x_0=0, v_0=0.2 \text{ m/s}, \ F=23 \text{ N}, \ \omega=2\omega_n.\)

\[
\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000 \text{ N/m}}{10 \text{ kg}}} = 10 \text{ rad/s}, \ \omega = 2\omega_n = 20 \text{ rad/s}
\]

\[
f_0 = \frac{F}{m} = \frac{23 \text{ N}}{10 \text{ kg}} = 2.3 \text{ N/kg}, \quad \frac{v_0}{\omega_n} = \frac{0.2 \text{ m/s}}{10 \text{ rad/s}} = 0.02 \text{ m}
\]

\[
\frac{f_0}{\omega_n^2 - \omega^2} = \frac{2.3 \text{ N/kg}}{(10^2 - 20^2) \text{ rad}^2 / \text{s}^2} = -7.9667 \times 10^{-3} \text{ m}
\]

Equation (2.11) then yields:

\[
x(t) = 0.02 \sin 10t + 7.667 \times 10^{-3} (\cos 10t - \cos 20t)
\]
Example 2.1.2 Given zero initial conditions a harmonic input of 10 Hz with 20 N magnitude and $k=2000\ \text{N/m}$, and measured response amplitude of 0.1 m, compute the mass of the system.

\[ x(t) = \frac{f_0}{\omega_n^2 - \omega^2} \left( \cos 20\pi t - \cos \omega_n t \right) \text{ for zero initial conditions} \]

trig identity $\Rightarrow x(t) = \frac{2f_0}{\omega_n^2 - \omega^2} \sin \left( \frac{\omega_n - \omega}{2} t \right) \sin \left( \frac{\omega_n + \omega}{2} t \right)$

$\Rightarrow \frac{2f_0}{\omega_n^2 - \omega^2} = 0.1 \Rightarrow \frac{2(20/\text{m})}{(2000/\text{m}) - (20\pi)^2} = 0.1$

\[ m = 0.45\ \text{kg} \]
Example 2.1.3 Design a rectangular mount for a security camera.

Compute $\ell$ so that the mount keeps the camera from vibrating more than 0.01 m of maximum amplitude under a wind load of 15 N at 10 Hz. The mass of the camera is 3 kg.
Solution: Modeling the mount and camera as a beam with a tip mass, and the wind as harmonic, the equation of motion becomes:

\[ m\ddot{x} + \frac{3EI}{l^3} x(t) = F_0 \cos \omega t \]

From strength of materials:

\[ I = \frac{bh^3}{12} \]

Thus the frequency expression is:

\[ \omega_n^2 = \frac{3Ebh^3}{12ml^3} = \frac{Ebh^3}{14ml^3} \]

Here we are interested computing \( l \) that will make the amplitude less than 0.01m:

\[ \left| \frac{2f_0}{\omega_n^2 - \omega^2} \right| < 0.01 \Rightarrow \begin{cases} 
(a) & -0.01 < \frac{2f_0}{\omega_n^2 - \omega^2}, \text{ for } \omega_n^2 - \omega^2 < 0 \\
(b) & \frac{2f_0}{\omega_n^2 - \omega^2} < 0.01, \text{ for } \omega_n^2 - \omega^2 > 0
\end{cases} \]
Case (a) (assume aluminum for the material):

\[-0.01 < \frac{2f_0}{\omega_n^2 - \omega^2} \Rightarrow 2f_0 < 0.01\omega_n^2 - 0.01\omega^2 \Rightarrow 0.01\omega^2 - 2f_0 > 0.01\frac{Ebh^3}{4m\ell^3}\]

\[\Rightarrow \ell^3 > 0.01\frac{Ebh^3}{4m(0.01\omega^2 - 2f_0)} = 0.321 \Rightarrow \ell > 0.6848 \text{ m}\]

Case (b):

\[\frac{2f_0}{\omega_n^2 - \omega^2} < 0.01 \Rightarrow 2f_0 < 0.01\omega_n^2 - 0.01\omega^2 \Rightarrow 2f_0 + 0.01\omega^2 < 0.01\frac{Ebh^3}{4m\ell^3}\]

\[\Rightarrow \ell^3 < 0.01\frac{Ebh^3}{4m(2f_0 + 0.01\omega^2)} = 0.191 \Rightarrow \ell < 0.576 \text{ m}\]
Remembering the constraint that the length must be at least 0.5 m, (a) and (b) yield

$$0.5 < \ell < 0.576, \text{ or } \ell > 0.6848 \text{ m}$$

Less material is usually desired, so chose case a, say $\ell = 0.55 \text{ m}$. 

To check, note that

$$\omega_n^2 - \omega^2 = \frac{3Efh^3}{12ml^3} - (20\pi)^2 = 1742 > 0$$

Thus the case a condition is met.

Next check the mass of the designed beam to insure it does not change the frequency. Note it is much less then $m$.

$$m = \rho l bh^3$$

$$= (2.7 \times 10^3)(0.55)(0.02)(0.02)^3$$

$$= 2.376 \times 10^{-4} \text{ kg}$$
A harmonic force may also be represented by sine or a complex exponential. How does this change the solution?

\[ m\ddot{x}(t) + kx(t) = F_0 \sin \omega t \quad \text{or} \quad \ddot{x}(t) + \omega_n^2 x(t) = f_0 \sin \omega t \quad (2.18) \]

The particular solution then becomes a sine:

\[ x_p(t) = X \sin \omega t \quad (2.19) \]

Substitution of (2.19) into (2.18) yields:

\[ x_p(t) = \frac{f_0}{\omega_n^2 - \omega^2} \sin \omega t \]

Solving for the homogenous solution and evaluating the constants yields

\[ x(t) = x_0 \cos \omega_n t + \left( \frac{v_0}{\omega_n} - \frac{\omega}{\omega_n} \frac{f_0}{\omega_n^2 - \omega^2} \right) \sin \omega_n t + \frac{f_0}{\omega_n^2 - \omega^2} \sin \omega t \quad (2.25) \]
Section 2.2 Harmonic Excitation of Damped Systems

Extending resonance and response calculation to damped systems
2.2 Harmonic excitation of damped systems

\[ m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F_0 \cos \omega t \]

\[ \ddot{x}(t) + 2\zeta \omega_n \dot{x}(t) + \omega_n^2 x(t) = f_0 \cos \omega t \]

\[ x_p(t) = X \cos(\omega t + \theta) \]

now includes a phase shift
Let $x_p$ have the form:

$$x_p(t) = A_s \cos \omega t + B_s \sin \omega t$$

$$X = \sqrt{A_s^2 + B_s^2}, \quad \theta = \tan^{-1}\left(\frac{B_s}{A_s}\right)$$

$$\dot{x}_p = -\omega A_s \sin \omega t + \omega B_s \cos \omega t$$

$$\ddot{x}_p = -\omega^2 A_s \cos \omega t - \omega^2 B_s \sin \omega t$$

Note that we are using the rectangular form, but we could use one of the other forms of the solution.
Substitute into the equations of motion

\[ (-\omega^2 A_s + 2\zeta \omega_n \omega B_s + \omega_n^2 A_s - f_0) \cos \omega t \]
\[ + \left( -\omega^2 B_s + 2\zeta \omega_n \omega A_s + \omega_n^2 B_s \right) \sin \omega t = 0 \]

for all time. Specifically for \( t = 0, 2\pi / \omega \Rightarrow \)

\[ (\omega_n^2 - \omega^2) A_s + (2\zeta \omega_n \omega) B_s = f_0 \]
\[ (-2\zeta \omega_n \omega) A_s + (\omega_n^2 - \omega^2) B_s = 0 \]
Write as a matrix equation:

\[
\begin{bmatrix}
(\omega_n^2 - \omega^2) & 2\zeta\omega_n\omega \\
-2\zeta\omega_n\omega & (\omega_n^2 - \omega^2)
\end{bmatrix}
\begin{bmatrix}
A_s \\
B_s
\end{bmatrix}
= \begin{bmatrix}
f_0 \\
0
\end{bmatrix}
\]

Solving for \(A_s\) and \(B_s\):

\[
A_s = \frac{(\omega_n^2 - \omega^2)f_0}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}
\]

\[
B_s = \frac{2\zeta\omega_n\omega f_0}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}
\]
Substitute the values of $A_s$ and $B_s$ into $x_p$:

$$x_p(t) = \frac{f_0}{\sqrt{\left(\omega_n^2 - \omega^2\right)^2 + (2\zeta\omega_n\omega)^2}} \cos(\omega t - \tan^{-1}\left(\frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2}\right))$$

Add homogeneous and particular to get total solution:

$$x(t) = Ae^{-\zeta\omega_n t} \sin(\omega_d t + \phi) + X \cos(\omega t - \theta)$$

- homogeneous or transient solution
- particular or steady state solution

Note: that $A$ and $\phi$ will not have the same values as in Ch 1, for the free response. Also as $t$ gets large, transient dies out.
Things to notice about damped forced response

• If $\zeta = 0$, undamped equations result
• Steady state solution prevails for large $t$
• Often we ignore the transient term (how large is $\zeta$, how long is $t$?)
• Coefficients of transient terms (constants of integration) are effected by the initial conditions AND the forcing function
• For underdamped systems at resonance the amplitude is finite.
Example 2.2.1: $\omega_n = 10 \text{ rad/s}$, $\omega = 5 \text{ rad/s}$, $\zeta = 0.01$, $F_0 = 1000 \text{ N}$, $m = 100 \text{ kg}$, and the initial conditions $x_0 = 0.05 \text{ m}$ and $v_0 = 0$. Compare $A$ and $f$ for forced and unforced case:

Using the equations on slide 6:

$$X = 0.133, \theta = -0.013$$

$$x(t) = Ae^{-0.1t} \sin(9.99t + \phi) + 0.133\cos(5t - 0.013)$$

Differentiating yields:

$$v(t) = -0.01Ae^{-0.1t} \sin(9.999t + \phi)$$

$$+ 9.999Ae^{-0.1t} \cos(9.999t + \phi)$$

$$- 0.665 \sin(5t - 0.013)$$

applying the intial conditions:

$$A = 0.519 \ (0.05), \ \phi = 0.875 \ (1.561)$$

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Proceeding with ignoring the transient

- Always check to make sure the transient is not significant
- For example, transients are very important in earthquakes
- However, in many machine applications transients may be ignored
Proceeding with ignoring the transient

Magnitude:

$$X = \frac{f_0}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta \omega_n \omega)^2}}$$  \hspace{0.5cm} (2.39)

Frequency ratio:

$$r = \frac{\omega}{\omega_n}$$

Non dimensional Form:

$$\frac{Xk}{F_0} = \frac{X \omega_n^2}{f_0} = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$  \hspace{0.5cm} (2.40)

Phase:

$$\theta = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right)$$
Magnitude plot

- Resonance is close to $r = 1$
- For $\zeta = 0$, $r = 1$ defines resonance
- As $\zeta$ grows resonance moves $r < 1$, and $X$ decreases
- The exact value of $r$, can be found from differentiating the magnitude

$$X = \frac{1}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$$

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Phase plot

- Resonance occurs at $\phi = \pi/2$
- The phase changes more rapidly when the damping is small
- From low to high values of $r$ the phase always changes by $180^0$ or $\pi$ radians

$$\theta = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right)$$
Example 2.2.3 Compute max peak by differentiating:

\[
\frac{d}{dr} \left( \frac{Xk}{F_0} \right) = \frac{d}{dr} \left( \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \right) = 0 \Rightarrow
\]

\[
r_{\text{peak}} = \sqrt{1-2\zeta^2} < 1 \Rightarrow \zeta < 1/\sqrt{2} \quad (2.41)
\]

\[
\left( \frac{Xk}{F_0} \right)_{\text{max}} = \frac{1}{2\zeta \sqrt{1-\zeta^2}} \quad (2.42)
\]
Effect of Damping on Peak Value

- The top plot shows how the peak value becomes very large when the damping level is small.
- The lower plot shows how the frequency at which the peak value occurs reduces with increased damping.
- Note that the peak value is only defined for values $\zeta < 0.707$.
Section 2.3 Alternative Representations

• A variety methods for solving differential equations
• So far, we used the method of undetermined coefficients
• Now we look at 3 alternatives:
  a geometric approach
  a frequency response approach
  a transform approach
• These also give us some insight and additional useful tools.
2.3.1 Geometric Approach

- Position, velocity and acceleration phase shifted each by $\pi/2$
- Therefore write each as a vector
- Compute $X$ in terms of $F_0$ via vector addition
Using vector addition on the diagram:

\[ F_0^2 = (k - m\omega^2)^2 X^2 + (c\omega)^2 X^2 \]

\[ X = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} \]

At resonance:

\[ \theta = \frac{\pi}{2}, \quad X = \frac{F_0}{c\omega} \]
2.3.2 Complex response method

\[ A e^{j\omega t} = A \cos \omega t + (A \sin \omega t) j \]  \hspace{1cm} (2.47)

\[ m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F_0 e^{j\omega t} \] \hspace{1cm} (2.48)

Real part of this complex solution corresponds to the physical solution
Choose complex exponential as a solution

\[ x_p(t) = X e^{j\omega t} \quad (2.49) \]

\[ (-\omega^2 m + cj\omega + k) X e^{j\omega t} = F_0 e^{j\omega t} \quad (2.50) \]

\[ X = \frac{F_0}{(k - m\omega^2) + (c\omega) j} = H(j\omega)F_0 \quad (2.51) \]

\[ H(j\omega) = \frac{1}{(k - m\omega^2) + (c\omega) j} \quad (2.52) \]

Note: These are all complex functions
Using complex arithmetic:

\[ X = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} e^{-j\theta} \]  

(2.53)

\[ \theta = \tan^{-1}\left(\frac{c\omega}{k - m\omega^2}\right) \]  

(2.54)

\[ x_p(t) = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} e^{j(\omega t - \theta)} \]  

(2.55)

Has real part = to previous solution
Comments:

• Label $x$-axis $\text{Re}(e^{j\omega t})$ and $y$-axis $\text{Im}(e^{j\omega t})$ results in the graphical approach

• It is the real part of this complex solution that is physical

• The approach is useful in more complicated problems
Example 2.3.1: Use the frequency response approach to compute the particular solution of an undamped system

The equation of motion is written as

\[ m\ddot{x}(t) + kx(t) = F_0 e^{j\omega t} \Rightarrow \ddot{x}(t) + \omega_n^2 x(t) = f_0 e^{j\omega t} \]

Let \( x_p(t) = X e^{j\omega t} \)

\[ \Rightarrow \left(-\omega^2 + \omega_n^2\right) X e^{j\omega t} = f_0 e^{j\omega t} \]

\[ \Rightarrow X = \frac{f_0}{\left(\omega_n^2 - \omega^2\right)} \]
2.3.3 Transfer Function Method

The Laplace Transform

- Changes ODE into algebraic equation
- Solve algebraic equation then compute the inverse transform
- Rule and table based in many cases
- Is used extensively in control analysis to examine the response
- Related to the frequency response function
The Laplace Transform approach:

- See appendix B and section 3.4 for details
- Transforms the time variable into an algebraic, complex variable
- Transforms differential equations into an algebraic equation
- Related to the frequency response method

\[ X(s) = \mathcal{L}(x(t)) = \int_{0}^{\infty} x(t)e^{-st} \, dt \]
Take the transform of the equation of motion:

\[ m\ddot{x} + c\dot{x} + kx = F_0 \cos \omega t \quad \Rightarrow \]

\[ (ms^2 + cs + k)X(s) = \frac{F_0s}{s^2 + \omega^2} \]

Now solve algebraic equation in \( s \) for \( X(s) \)

\[ X(s) = \frac{F_0s}{(ms^2 + cs + k)(s^2 + \omega^2)} \]

To get the time response this must be “inverse transformed”
Transfer Function Method

With zero initial conditions:

\[(ms^2 + cs + k)X(s) = F(s) \Rightarrow\]

\[
\frac{X(s)}{F(s)} = H(s) = \frac{1}{ms^2 + cs + k}
\]

The transfer function

\[
H(j\omega) = \frac{1}{k - m\omega^2 + c\omega j}
\]

= frequency response function

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Example 2.3.2 Compute forced response of the suspension system shown using the Laplace transform

Summing moments about the shaft:

\[ J \ddot{\theta}(t) + k \theta(t) = a F_0 \sin \omega t \]

Taking the Laplace transform:

\[ Js^2 X(s) + kX(s) = a F_0 \frac{\omega}{s^2 + \omega^2} \]

\[ \Rightarrow X(s) = a \omega F_0 \frac{1}{(s^2 + \omega^2)(Js^2 + k)} \]

Taking the inverse Laplace transform:

\[ \theta(t) = a \omega F_0 L^{-1} \left( \frac{1}{(s^2 + \omega^2)(Js^2 + k)} \right) \]

\[ \Rightarrow \theta(t) = \frac{a \omega F}{J} \frac{1}{\omega^2 - \omega_n^2} \left( \frac{1}{\omega} \sin \omega t - \frac{1}{\omega_n} \sin \omega_n t \right), \quad \omega_n = \sqrt{\frac{k}{J}} \]
Notes on Phase for Homogeneous and Particular Solutions

• Equation (2.37) gives the full solution for a harmonically driven underdamped SDOF oscillator to be

\[ x(t) = Ae^{-\zeta \omega_n t} \sin(\omega_d t + \theta) + X \cos(\omega t - \phi) \]

How do we interpret these phase angles? Why is one added and the other subtracted?
Non-Zero initial conditions

\[ x_0 \neq 0 \quad \nu_0 \neq 0 \]

\[ \sin(\omega_d t + \theta) \]

\[ \theta = \tan^{-1} \left( \frac{x_o \omega_d}{\nu_o + \zeta \omega_n x_o} \right) \]
Zero initial displacement

\[ x_0 = 0 \quad v_0 \neq 0 \]

\[ \theta = \tan^{-1} \left( \frac{x_o \omega_d}{v_o + \zeta \omega_n x_o} \right) = \tan^{-1} \left( \frac{0}{v_o} \right) = 0 \]

\[ \sin(\omega_d t + \theta) \]
Zero initial velocity

\[ x_0 \neq 0 \quad v_0 = 0 \]

\[ \theta = \tan^{-1} \left( \frac{x_0 \omega_d}{v_o + \zeta \omega_n x_o} \right) = \frac{\sqrt{1 - \zeta^2}}{\zeta} \]

\[ \sin(\omega_d t + \theta) \]
Phase on Particular Solution

\[ x_p(t) = X \cos(\omega t - \phi) \]

- Simple “atan” gives \(-\pi/2 < \phi < \pi/2\)
- Four-quadrant “atan2” gives \(0 < \phi < \pi\)