Section 1.8 Stability

Stability is defined for the solution of free response case:

Stable: \[ |x(t)| < M, \quad \forall \; t > 0 \]

Asymptotically Stable: \[ \lim_{t \to \infty} x(t) = 0 \]

Unstable:

if it is not stable or asymptotically stable
Examples of the types of stability

Stable

Asymptotically Stable

Divergent instability

Flutter instability

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Example: 1.8.1: For what values of the spring constant will the response be stable?

Figure 1.37

\[ m\ell^2\ddot{\theta} + \left(\frac{k\ell^2}{2}\sin\theta\right)\cos\theta - mg\ell\sin\theta = 0 \Rightarrow m\ell^2\ddot{\theta} + \frac{k\ell^2}{2}\theta - mg\ell\theta = 0 \]

\[ \Rightarrow 2m\ell\ddot{\theta} + (k\ell - 2mg)\theta = 0 \quad \text{(for small } \theta) \]

\[ \Rightarrow k > 2mg \quad \text{for a stable response} \]
1.9 Numerical Simulation

- Solving differential equations by numerical integration
- Euler, Runge-Kutta, etc.
- Available in Mathcad, Matlab, Mathematica and Maple (or in FORTRAN)
- Or use Engineering Vibration Toolbox
- Will use these to examine nonlinear vibration problems that do not have analytical expressions for solutions

\[ \frac{dx(t_i)}{dt} = \lim_{\Delta t \to 0} \frac{x(t_{i+1}) - x(t_i)}{\Delta t} \]

\[ \Delta t = t_{i+1} - t_i \]
First order differential equation

solve \( \dot{x}(t) = ax(t), \ x(0) = x_0 \)

\[
\frac{x_{i+1} - x_i}{\Delta t} = ax_i, \ x_0
\]

\[
x_i = x(t_i), \Delta t = t_{i+1} - t_i
\]

\[
x_{i+1} = x_i \left[1 + a\Delta t \right]
\]

The new value of \( x \) is calculated from the old value of \( x \).

\( x_3 \) will be used to calculate the next term \( x_4 \).
Example 9.7.1 \textbf{solve} \frac{dx}{dt}=-3x, \ x(0)=1

\begin{align*}
a = -3, \ \text{take} \ \Delta t = 0.5 \\
x_0 &= 1 \\
x_1 &= x_0 + (0.5)(-3)(x_0) = -0.5 \\
x_2 &= x_1 + (0.5)(-3)(x_1) = 0.25 \\
&\vdots \quad & = \vdots \\
x(t) &= Ae^{\lambda t} \\
\dot{x}(t) &= -3x(t) \implies -\lambda Ae^{\lambda t} = -3Ae^{\lambda t} \\
\implies \lambda &= 3, \\
x(0) &= x_0 = 1 = Ae^{0} \implies A = 1 \text{ so that} \\
x(t) &= e^{-3t}
\end{align*}

\textbf{Numerical solution} \\
Note that the numerical Solution is different for Each choice of $\Delta t$

\textbf{Analytical solution}
Time step

• With time step at 0.5 sec the numerical solution oscillates about the exact solution.

• Large errors can be caused by choosing the time step to be too small.

• Small time steps require more computation.

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Numerical solution of the 2nd order equation of vibration:

It is necessary to convert the second order equation into two first order equations. To achieve this two new variables $x_1$ and $x_2$ are defined as follows.

$$m\ddot{x} + c\dot{x} + kx = 0$$

Let $x_1 = x$, $x_2 = \dot{x}$

From this two first order differential equations can be written.

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{c}{m} x_2 - \frac{k}{m} x_1$$

Called state space
Matrix form

Combining these first order DEs in matrix form gives.

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
-k/m & -c/m
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]

The Euler numerical method can then be applied to the
matrix form to give.

\[
x(t_{i+1}) = x(t_i) + \Delta t A x(t_i)
\]

\[
x_{i+1} = [I + \Delta t A] x_i
\]
Matlab Solutions ‘ode23’ and ‘ode45’

- Use Runge-Kutta. More sophisticated than the Euler method but more accurate
- Often picks $\Delta t$ (i.e. if solution $x(t)$ is rapidly changing $\Delta t$ is chosen to be small and visa-versa
- Works for nonlinear equations too

Create Matlab function

```matlab
function xdot=sdof(t,x)
k=2;c=1;m=3;
A=[0 1;-k/m -c/m];
xdot=A*x;
```

In the command window

```matlab
» t0=0;tf=20;
» x0=[0 ; 0.25];
» [t,x]=ode45('sdof',[t0 tf],x0);
» plot(t,x)
```

Saved as sdof.m
Resulting solution
Why use numerical simulation when we can compute the analytical solution and plot it?

• To have a tool that we are confident with that will allow us to solve for the response when an analytical solution cannot be found

• Nonlinear systems to not have analytical solutions, but can be simulated numerically
Section 1.10 Coulomb Friction and the Pendulum

Nonlinear phenomenon in vibration analysis
Vibration of Nonlinear Systems

Sliding or Coulomb Friction

\[ f_c(\dot{x}) = \begin{cases} 
-\mu N & \dot{x}(t) > 0 \\
0 & \dot{x}(t) = 0 \\
\mu N & \dot{x}(t) < 0 
\end{cases} \]

The force due to Coulomb friction opposes motion, hence the ‘sgn’ function is used. The force is proportional to the normal force and independent of the velocity of the mass.
The free body diagram split depending on the direction of motion:

\[ W = mg \]

\[ f_c = \mu N = \mu mg \]

\[ kx \]

mass moving left

\[ \dot{x}(t) < 0 \]

mass moving right

\[ \dot{x}(t) > 0 \]

\[ \Rightarrow m\ddot{x} + \mu mg \text{ sgn}(\dot{x}) + kx = 0 \quad (1.92) \]
The ‘sgn’ function is nonlinear

- Causes equation of motion to be nonlinear
- Can solve as piecewise linear (see text)
- Can solve numerically
- Has more than one equilibrium position
- Decay is linear rather than exponential
- Comes to rest when spring cannot overcome friction at the instant the velocity is zero

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Figure 1.42 shows the details of the free response of a system with Coulomb damping
A General second order system can be written as a single first order equation

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix} x_2 \\ -f(x_1, x_2) \end{bmatrix} = F(x)
\]

The equilibrium position is defined:

\[F(x_e) = 0\]

For Coulomb friction this is defined as:

\[x_2 = 0 \text{ and } -\frac{\mu_s mg}{k} < x_1 < \frac{\mu_s mg}{k}\]

i.e. the positions where the force due to the spring can no longer overcome the sliding friction force
Example 1.10.2: Calculating the equilibrium position for nonlinear DEs

Equation of motion:
\[ \ddot{x} + x - \beta^2 x^3 = 0 \]

State space form:
\[ \dot{x}_1 = x_2 \]
\[ \dot{x}_2 = x_1 (\beta^2 x_1^2 - 1) \]

Equilibrium positions:
\[ x_2 = 0 \]
\[ x_1 (\beta^2 x_1^2 - 1) = 0 \Rightarrow \]
\n\[ x_e = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ \beta \end{bmatrix}, \begin{bmatrix} -1 \\ \beta \end{bmatrix} \]

Multiple equilibrium positions possible
The pendulum

Stable Equilibrium
\[ \theta = 0, 2\pi, 4\pi \ldots \]

Unstable Equilibrium
\[ \theta = \pi, 3\pi, 5\pi \ldots \]
Example 1.10.2 Equilibrium of a Pendulum

Figure 1.44

\[ \ddot{\theta}(t) + \frac{g}{\ell} \sin \theta(t) = 0 \]

\[ x_1 = \theta, \quad x_2 = \dot{\theta} \]

\[ \Rightarrow \]

\[ \dot{x}_1 = x_2 \]

\[ \dot{x}_2 = -\frac{g}{\ell} \sin x_1 \]

\[ \Rightarrow \mathbf{F}(x) = \begin{bmatrix} x_2 \\ -\frac{g}{\ell} \sin x_1 \end{bmatrix} = 0 \]

\[ \Rightarrow x_2 = 0 \]

\[ \sin x_1 = 0 \]

\[ \Rightarrow x_2 = 0 \quad \text{and} \quad x_1 = n\pi, \quad n = 0, 1, 2 \ldots \]
Solution to the pendulum

- Can use numerical simulation to examine both linear and nonlinear response
- Let \( (g/L) = (0.1)^2 \) so that \( \omega_n = 0.1 \)
- a) use \( \theta(0) = 0.3 \) rad & initial vel: 0.3 rad/s
- b) change the initial position to: \( \theta(0) = \pi \) rad which is near the unstable equilibrium

\[ 
\begin{align*}
\theta(0) &= 0.3 \text{ rad} & \text{Initial velocity} &= 0.3 \text{ rad/s} \\
\theta(0) &= \pi \text{ rad} 
\end{align*}
\]
Pendulum with friction added

\[ \ddot{\theta} + c \dot{\theta} + \frac{g}{\ell} \sin \theta = 0 \]

After making a single loop the pendulum cannot make a second rotation and settles to the stable equilibrium position of \( \theta = 4\pi \).

Friction loss causes slow decay.
Summary of Nonlinear Vibrations

- Additional phenomena over linear case
- Multiple equilibrium
- Instabilities possible with positive coefficients
- Form of response dependent on initial conditions
- Closed form solutions usually not available
- Can simulate numerically
- Linear model has tremendous advantages
- Linear combination of inputs yields linear combination of outputs
- Linear ode techniques very powerful
- But don’t make a design error by ignoring important nonlinear situations
- All systems have nonlinear ranges of operation
- Need to sort out when nonlinearity is important to consider and when to ignore it

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