Section 1.4 Modeling and Energy Methods

• An alternative way to determine the equation of motion and an alternative way to calculate the natural frequency of a system

• Useful if the forces or torques acting on the object or mechanical part are difficult to determine
Potential and Kinetic Energy

The potential energy of mechanical systems $U$ is often stored in “springs” (remember that for a spring $F=kx$)

$$U_{spring} = \int_0^{x_0} F \, dx = \int_0^{x_0} kx \, dx = \frac{1}{2} kx_0^2$$

The kinetic energy of mechanical systems $T$ is due to the motion of the “mass” in the system

$$T_{trans} = \frac{1}{2} mx^2, \quad T_{rot} = \frac{1}{2} J \dot{\theta}^2$$
Conservation of Energy

For a simple, conservative (i.e. no damper), mass spring system the energy must be conserved:

\[ T + U = \text{constant} \]

or

\[ \frac{d}{dt}(T + U) = 0 \]

At two different times \( t_1 \) and \( t_2 \) the increase in potential energy must be equal to a decrease in kinetic energy (or visa-versa).

\[ U_1 - U_2 = T_2 - T_1 \]

and

\[ U_{\text{max}} = T_{\text{max}} \]
Deriving the equation of motion from the energy

\[ \frac{d}{dt} (T + U) = \frac{d}{dt} \left( \frac{1}{2} m \ddot{x}^2 + \frac{1}{2} k x^2 \right) = 0 \]

\[ \Rightarrow \dot{x} (m \ddot{x} + kx) = 0 \]

Since \( \dot{x} \) cannot be zero for all time, then

\[ m \ddot{x} + kx = 0 \]
Determining the Natural frequency directly from the energy

If the solution is given by $x(t) = A\sin(\omega t + \phi)$ then the maximum potential and kinetic energies can be used to calculate the natural frequency of the system

$$U_{\text{max}} = \frac{1}{2}kA^2 \quad T_{\text{max}} = \frac{1}{2}m(\omega_n A)^2$$

Since these two values must be equal

$$\frac{1}{2}kA^2 = \frac{1}{2}m(\omega_n A)^2$$

$$\Rightarrow k = m\omega_n^2 \Rightarrow \omega_n = \sqrt{\frac{k}{m}}$$
Example 1.4.1

Compute the natural frequency of this roller fixed in place by a spring. Assume it is a conservative system (i.e. no losses) and rolls without slipping.

\[ T_{\text{rot}} = \frac{1}{2} J \dot{\theta}^2 \quad \text{and} \quad T_{\text{trans}} = \frac{1}{2} m \dot{x}^2 \]
Solution continued

\[ x = r \theta \Rightarrow \dot{x} = r \dot{\theta} \Rightarrow T_{\text{Rot}} = \frac{1}{2} J \frac{\dot{x}^2}{r^2} \]

The max value of \( T \) happens at \( v_{\text{max}} = \omega_n A \)

\[ \Rightarrow T_{\text{max}} = \frac{1}{2} J \left( \frac{\omega_n A}{r^2} \right)^2 + \frac{1}{2} m (\omega_n A)^2 = \frac{1}{2} \left( m + \frac{J}{r^2} \right) \omega_n^2 A^2 \]

The max value of \( U \) happens at \( x_{\text{max}} = A \)

\[ \Rightarrow U_{\text{max}} = \frac{1}{2} k A^2 \quad \text{Thus} \quad T_{\text{max}} = U_{\text{max}} \quad \Rightarrow \]

\[ \frac{1}{2} \left( m + \frac{J}{r^2} \right) \omega_n^2 A^2 = \frac{1}{2} k A^2 \Rightarrow \omega_n = \sqrt{\frac{k}{m + \frac{J}{r^2}}} \]

Effective mass
Example 1.4.2 Determine the equation of motion of the pendulum using energy

\[ J = m\ell^2 \]

\[ \theta \]

\[ mg \]
Now write down the energy

\[
T = \frac{1}{2} J_0 \dot{\theta}^2 = \frac{1}{2} m \ell^2 \dot{\theta}^2
\]

\[
U = mg \ell (1 - \cos \theta), \quad \text{the change in elevation is } \ell (1 - \cos \theta)
\]

\[
\frac{d}{dt} (T + U) = \frac{d}{dt} \left( \frac{1}{2} m \ell^2 \dot{\theta}^2 + mg \ell (1 - \cos \theta) \right) = 0
\]
\[ m\ell^2 \ddot{\theta} + mg \ell (\sin \theta) \dot{\theta} = 0 \]
\[ \Rightarrow \dot{\theta} \left( m\ell^2 \ddot{\theta} + mg \ell (\sin \theta) \right) = 0 \]
\[ \Rightarrow m\ell^2 \ddot{\theta} + mg \ell (\sin \theta) = 0 \]
\[ \Rightarrow \ddot{\theta}(t) + \frac{g}{\ell} \sin \theta(t) = 0 \]
\[ \Rightarrow \ddot{\theta}(t) + \frac{g}{\ell} \theta(t) = 0 \quad \Rightarrow \omega_n = \sqrt{\frac{g}{\ell}} \]
Example 1.4.4  The effect of including the mass of the spring on the value of the frequency.
mass of element $dy: \frac{m_s}{\ell} \, dy$

velocity of element $dy: v_{dy} = \frac{y}{\ell} \dot{x}(t),$

$T_{spring} = \frac{1}{2} \int_{0}^{\ell} \frac{m_s}{\ell} \left[ \frac{y}{\ell} \dot{x} \right]^2 \, dy$ (adds up the KE of each element)

$= \frac{1}{2} \left( \frac{m_s}{3} \right) \dot{x}^2$

$T_{mass} = \frac{1}{2} m \dot{x}^2 \Rightarrow T_{tot} = \left[ \frac{1}{2} \left( \frac{m_s}{3} \right) + \frac{1}{2} m \right] \dot{x}^2 \Rightarrow T_{max} = \frac{1}{2} \left( m + \frac{m_s}{3} \right) \omega_n^2 A^2$

$U_{max} = \frac{1}{2} kA^2$

$\Rightarrow \omega_n = \sqrt{\frac{k}{m + \frac{m_s}{3}}}$

• This provides some simple design and modeling guides
What about gravity?

\[ mg - k\Delta = 0, \text{ from FBD, and static equilibrium} \]

\[ U_{\text{spring}} = \frac{1}{2} k(\Delta + x)^2 \]

\[ U_{\text{grav}} = -mgx \]

\[ T = \frac{1}{2} m\dot{x}^2 \]
Now use $\frac{d}{dt}(T + U) = 0$

$$\Rightarrow \frac{d}{dt}\left[ \frac{1}{2}m\ddot{x}^2 - mgx + \frac{1}{2}k(\Delta + x)^2 \right] = 0$$

$$\Rightarrow m\dddot{x} - mg\dot{x} + k(\Delta + x)\dot{x}$$

$$\Rightarrow \dot{x}(m\dddot{x} + kx) + \dot{x}(k\Delta - mg) = 0$$

$$\Rightarrow m\dddot{x} + kx = 0$$

*Gravity does not effect the equation of motion or the natural frequency of the system for a linear system as shown previously with a force balance.*
Lagrange’s Method for deriving equations of motion.

Again consider a conservative system and its energy.

It can be shown that if the Lagrangian $L$ is defined as

\[ L = T - U \]

Then the equations of motion can be calculated from

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0 \]  \hspace{1cm} (1.63)

Which becomes

\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}} \right) - \frac{\partial T}{\partial q} + \frac{\partial U}{\partial q} = 0 \]  \hspace{1cm} (1.64)

Here $q$ is a generalized coordinate.
Example 1.4.7 Derive the equation of motion of a spring mass system via the Lagrangian

\[ T = \frac{1}{2} m \dot{x}^2 \quad \text{and} \quad U = \frac{1}{2} kx^2 \]

Here \( q = x \), and and the Lagrangian becomes

\[ L = T - U = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} kx^2 \]

Equation (1.64) becomes

\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{x}} \right) - \frac{\partial T}{\partial x} + \frac{\partial U}{\partial x} = \frac{d}{dt} \left( m \dot{x} \right) - 0 + kx = 0 \]

\[ \Rightarrow m \ddot{x} + kx = 0 \]
1.5 More on springs and stiffness

- Longitudinal motion
- $A$ is the cross sectional area ($m^2$)
- $E$ is the elastic modulus (Pa=N/m$^2$)
- $\ell$ is the length (m)
- $k$ is the stiffness (N/m)
Figure 1.21 Torsional Stiffness

- $J_p$ is the polar moment of inertia of the rod
- $J$ is the mass moment of inertia of the disk
- $G$ is the shear modulus, $\ell$ is the length
Example 1.5.1 compute the frequency of a shaft/mass system \( \{J = 0.5 \text{ kg m}^2\} \)

From Equation (1.50)

\[
\sum M = J \ddot{\theta} \Rightarrow J \ddot{\theta}(t) + k \theta(t) = 0
\]

\[
\Rightarrow \ddot{\theta}(t) + \frac{k}{J} \theta(t) = 0
\]

\[
\Rightarrow \omega_n = \sqrt{\frac{k}{J}} = \sqrt{\frac{GJ_p}{\ell J}}, \quad J_p = \frac{\pi d^4}{32}
\]

For a 2 m steel shaft, diameter of 0.5 cm ⇒

\[
\omega_n = \sqrt{\frac{GJ_p}{\ell J}} = \sqrt{\frac{(8 \times 10^{10} \text{ N/m}^2)[\pi (0.5 \times 10^{-2} \text{ m})^4 / 32]}{(2 \text{ m})(0.5 \text{ kg} \cdot \text{m}^2)}}
\]

\[
= 2.2 \text{ rad/s}
\]
Fig. 1.22 Helical Spring

\[ d = \text{diameter of wire} \]
\[ 2R = \text{diameter of turns} \]
\[ n = \text{number of turns} \]
\[ x(t) = \text{end deflection} \]
\[ G = \text{shear modulus of spring material} \]

\[ k = \frac{Gd^4}{64nR^3} \]

Allows the design of springs to have specific stiffness.
Fig 1.23 Transverse beam stiffness

- Strength of materials and experiments yield:

\[ k = \frac{3EI}{l^3} \]

With a mass at the tip:

\[ \omega_n = \sqrt{\frac{3EI}{m \cdot l^3}} \]
Samples of Vibrating Systems

• Deflection of continuum (beams, plates, bars, etc) such as airplane wings, truck chassis, disc drives, circuit boards…
• Shaft rotation
• Rolling ships
• See text for more examples.
Example 1.5.2 Effect of fuel on frequency of an airplane wing

- Model wing as transverse beam
- Model fuel as tip mass
- Ignore the mass of the wing and see how the frequency of the system changes as the fuel is used up
Mass of pod 10 kg empty 1000 kg full
\( \ell = 5.2 \times 10^{-5} \text{ m}^4, \ E = 6.9 \times 10^9 \text{ N/m}, \ \ell = 2 \text{ m} \)

- Hence the natural frequency changes by an order of magnitude while it empties out fuel.

\[
\omega_{\text{full}} = \sqrt{\frac{3EI}{ml^3}} = \sqrt{\frac{3(6.9 \times 10^9)(5.2 \times 10^{-5})}{1000 \cdot 2^3}} = 11.6 \text{ rad/s} = 1.8 \text{ Hz}
\]

\[
\omega_{\text{empty}} = \sqrt{\frac{3EI}{ml^3}} = \sqrt{\frac{3(6.9 \times 10^9)(5.2 \times 10^{-5})}{10 \cdot 2^3}} = 115 \text{ rad/s} = 18.5 \text{ Hz}
\]

This ignores the mass of the wing
Example 1.5.3 Rolling motion of a ship

\[ J \ddot{\theta}(t) = -W \bar{G}Z = -Wh \sin \theta(t) \]

For small angles this becomes

\[ J \ddot{\theta}(t) + Wh\dot{\theta}(t) = 0 \]

\[ \Rightarrow \omega_n = \sqrt{\frac{hW}{J}} \]
Combining Springs: Springs are usually only available in limited stiffness values. Combing them allows other values to be obtained.

- Equivalent Spring

  series: \( k_{AC} = \frac{1}{\frac{1}{k_1} + \frac{1}{k_2}} \)

  parallel: \( k_{ab} = k_1 + k_2 \)

This is identical to the combination of capacitors in electrical circuits.
Use these to design from available parts

- Discrete springs available in standard values
- Dynamic requirements require specific frequencies
- Mass is often fixed or ± small amount
- Use spring combinations to adjust $w_n$
- Check static deflection
Example 1.5.5  Design of a spring mass system
using available springs: series vs parallel

- Let $m = 10$ kg
- Compare a series and parallel combination
  - a) $k_1 = 1000$ N/m, $k_2 = 3000$ N/m, $k_3 = k_4 = 0$
  - b) $k_3 = 1000$ N/m, $k_4 = 3000$ N/m, $k_1 = k_2 = 0$
Case a) parallel connection:

\[ k_3 = k_4 = 0, k_{eq} = k_1 + k_2 = 1000 + 3000 = 4000 \text{ N/m} \]

\[ \Rightarrow \omega_{parallel} = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{4000}{10}} = 20 \text{ rad/s} \]

Case b) series connection:

\[ k_1 = k_2 = 0, k_{eq} = \frac{1}{(1/k_3) + (1/k_4)} = \frac{3000}{3+1} = 750 \text{ N/m} \]

\[ \Rightarrow \omega_{series} = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{750}{10}} = 8.66 \text{ rad/s} \]

Same physical components, very different frequency
Allows some design flexibility in using off the shelf components
Example: Find the equivalent stiffness $k$ of the following system (Fig 1.26, page 47)

$$\omega_n = \sqrt{\frac{k_1 k_3 + k_2 k_3 + k_5 k_3 + k_1 k_4 + k_2 k_4 + k_5 k_4 + k_3 k_4}{m(k_3 + k_4)}}$$
Example 1.5.5  Compare the natural frequency of two springs connected to a mass in parallel with two in series

A series connect of \( k_1 = 1000 \text{ N/m} \) and \( k_2 = 3000 \text{ N/m} \) with \( m = 10 \text{ kg} \) yields:

\[
 k_{eq} = \frac{1}{1/1000 + 1/3000} = 750 \text{ N/m} \Rightarrow \omega_{series} = \sqrt{\frac{750 \text{ N/m}}{10 \text{ kg}}} = 8.66 \text{ rad/s}
\]

A parallel connect of \( k_1 = 1000 \text{ N/m} \) and \( k_2 = 3000 \text{ N/m} \) with \( m = 10 \text{ kg} \) yields:

\[
 k_{eq} = 1000 \text{ N/m} + 3000 \text{ N/m} = 4000 \text{ N/m} \Rightarrow \omega_{par} = \sqrt{\frac{4000 \text{ N/m}}{10 \text{ kg}}} = 20 \text{ rad/s}
\]

Same components, very different frequency
Static Deflection

Another important consideration in designing with springs is the static deflection

\[ \Delta k = mg \implies \Delta = \frac{mg}{k} \]

This determines how much a spring compresses or sags due to the static mass (you can see this when you jack your car up)

The other concern is “rattle space” which is the maximum deflection \( A \)
Section 1.6 Measurement

- **Mass**: usually pretty easy to measure using a balance— a static experiment
- **Stiffness**: again can be measured statically by using a simple displacement measurement and knowing the applied force
- **Damping**: can only be measured dynamically
Measuring moments of inertia using a Trifilar suspension system

\[ J = \frac{gT^2 r_0^2 (m_0 + m)}{4\pi^2} - J_0 \]

\( T \) is the measured period
\( g \) is the acceleration due to gravity
Stiffness Measurements

From Static Deflection:

\[ F = k \times x \quad \text{or} \quad \sigma = E \varepsilon \]

\[ \Rightarrow k = \frac{F}{x} \]

From Dynamic Frequency:

\[ \omega_n = \sqrt{\frac{k}{m}} \quad \Rightarrow \quad k = m \omega_n^2 \]
Example 1.6.1 Use the beam stiffness equation to compute the modulus of a material

\[ E = \text{elastic modulus} \]
\[ l = \text{length of beam} \]
\[ I = \text{moment of inertia of cross-sectional area about the neutral axis} \]

Figure 1.24  \( l = 1 \text{ m}, m = 6 \text{ kg}, I = 10^{-9} \text{ m}^4 \), and measured \( T = 0.62 \text{ s} \)

\[
T = 2\pi \sqrt{\frac{ml^3}{3EI}} = 0.62 \text{ s}
\]

\[
\Rightarrow E = \frac{4\pi^2 ml^3}{3T^2I} = \frac{4\pi^2 (6 \text{ kg})(1 \text{ m})^3}{3(0.62 \text{ s})^2 \left(10^{-9} \text{ m}^4\right)} = 2.05 \times 10^{11} \text{ N/m}^2
\]
Damping Measurement (Dynamic only)

Define the Logarithmic Decrement:

\[
\delta = \ln \frac{x(t)}{x(t + T)} \tag{1.71}
\]

\[
\delta = \ln \frac{Ae^{-\zeta \omega_n t} \sin(\omega_d t + \phi)}{Ae^{-\zeta \omega_n (t + T)} \sin(\omega_d t + \omega_d T + \phi)} \tag{1.72}
\]

\[
\delta = \zeta \omega_n T
\]

\[
\zeta = \frac{c}{c_{cr}} = \frac{\delta}{\omega_n T} = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} \tag{1.75}
\]
Section 1.7: Design Considerations

Using the analysis so far to guide the selection of components.
Example 1.7.1

- Mass $2 \text{ kg} \leq m \leq 3 \text{ kg}$ and $k \geq 200 \text{ N/m}$
- For a possible frequency range of $8.16 \text{ rad/s} \leq \omega_n \leq 10 \text{ rad/s}$
- For initial conditions: $x_0 = 0$, $v_0 < 300 \text{ mm/s}$
- Choose a $c$ so response is always $\leq 25 \text{ mm}$
Solution:

- Write down $x(t)$ for 0 initial displacement
- Look for max amplitude
- Occurs at time of first peak ($T_{\max}$)
- Compute the amplitude at $T_{\max}$
- Compute $\zeta$ for $A(T_{\max}) = 0.025$
\[ x(t) = \frac{v_0}{\omega_n} e^{-\zeta \omega_n t} \sin(\omega_d t) \]

\[ \Rightarrow \text{worst case happens at smallest } \omega_d \Rightarrow \omega_n = 8.16 \text{ rad/s} \]

\[ \Rightarrow \text{worst case happens at max } v_0 = 300 \text{ mm/s} \]

With \( \omega_n \) and \( v_0 \) fixed at these values, investigate how \( v \) varies with \( \zeta \)

First peak is highest and occurs at

\[ \frac{d}{dt}(x(t)) = 0 \Rightarrow \omega_d e^{-\zeta \omega_n t} \cos(\omega_d t) - \zeta \omega_n e^{-\zeta \omega_n t} \sin(\omega_d t) = 0 \]

Solve for \( t = T_{\text{max}} \Rightarrow T_m = \frac{1}{\omega_d} \tan^{-1}(\frac{\omega_d}{\zeta \omega_n}) = \frac{1}{\omega_d} \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) \]

Sub \( T_{\text{max}} \) into \( x(t) \):

\[ A_m(\zeta) = x(T_m) = \frac{v_0}{\omega_n \sqrt{1-\zeta^2}} e^{-\zeta \tan^{-1}(\sqrt{1-\zeta^2})} \sin(\tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)) \]

\[ A_m(\zeta) = \frac{v_0}{\omega_n} e^{-\zeta \tan^{-1}(\sqrt{1-\zeta^2})} \]
To keep the max value less then 0.025 m solve
\[ A_{\text{max}} (\zeta) = 0.025 \Rightarrow \zeta = 0.281 \]
Using the upper limit on the mass \((m = 3 \text{ kg})\) yields
\[ c = 2m\omega_n\zeta = 2 \cdot 3 \cdot 8.16 \cdot 0.281 = 14.15 \text{ kg/s} \]

FYI, \(\zeta = 0\) yields \(A_{\text{max}} = \frac{v_0}{\omega_n} = 37 \text{ mm}\)
Example 1.7.3: What happens to a good design when some one changes the parameters? (Car suspension system). How does $\zeta$ change with mass?

Given $\zeta = 1$, $m = 1361$ kg, $\Delta = 0.05$ m, compute $c, k$.

$\omega_n = \sqrt{\frac{k}{m}} \Rightarrow k = 1361 \omega_n^2, \ mg = k\Delta \Rightarrow k = \frac{mg}{\Delta}$

$\Rightarrow \omega_n = \sqrt{\frac{mg}{m\Delta}} = \sqrt{\frac{9.81}{0.05}} = 14$ rad/s $\Rightarrow$

$k = 1361(14)^2 = 2.668 \times 10^5$ N/m

$\zeta = 1 \Rightarrow c = 2m\omega_n = 2(1361)(14) = 3.81 \times 10^4$ kg/s
Now add 290 kg of passengers and luggage. What happens?

\[ m = 1361 + 290 = 1651 \text{ kg} \]

\[ \Rightarrow \Delta = \frac{mg}{k} = \frac{1651 \cdot 9.8}{2.668 \times 10^5} \approx 0.06 \text{ m} \]

\[ \Rightarrow \omega_n = \sqrt{\frac{g}{\Delta}} = \sqrt{\frac{9.8}{0.06}} = 12.7 \text{ rad/s} \]

\[ \zeta = \frac{c}{c_{cr}} = \frac{3.81 \times 10^4}{2m \omega_n} = 0.9 \]

So some oscillation results at a lower frequency.