Flow Separation Control Using a Convection Based POD Approach

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The standard Proper Orthogonal Decomposition (POD) is an optimal tool to extract the energy-containing structures from a turbulent flow field. Some POD applications involve more than the structure identification process, and use the resulting eigenfunctions as a subspace onto which the flow state equations are projected, thus creating a low-dimensional model for the system under study. For these more elaborate applications, an increasing number of which are in the field of flow control, this low-dimensional plant is expected to represent the flow dynamics as accurately as possible. In this work, which ultimately intends to control the flow separation over an airfoil using a dynamical model of the flow, we introduce a variation to the POD formulation that we will refer to as the convection POD or cPOD. This formulation is developed using the non-linear convection terms of the Navier-Stokes equations to build a new kernel, thus producing a subspace knowledgeable about the dynamical realizations in the flow. We show that the convection POD succeeds in capturing the dynamical features of the flow more effectively than the standard formulation. It is found that the eigenfunctions now reveal physical structures in the flow field as opposed to patterns of highest energy concentration. Possible improvements in flow control applications and potential difficulties associated with this method are also discussed.

Nomenclature

\( a_n(t) \) POD mode n expansion coefficient
\( a_{n,POD}(t) \) cPOD mode n expansion coefficient
\( \tilde{a}_n(t) \) POD/mLSM mode n estimated expansion coefficient
\( \tilde{a}_{n,POD}(t) \) cPOD/mLSM mode n estimated expansion coefficient
\( A_n \) mLSM linear coefficient for POD mode n and \( i^{th} \) pressure sensor
\( A_{i,n}, B_{i,j} \) mQSM linear and quadratic coefficients for POD mode n
\( C(t, t') \) snapshot POD temporal correlation tensor
\( C_{cPOD}(t, t') \) snapshot cPOD temporal correlation tensor
\( M \) number of PIV velocity vectors
\( N \) total number of POD modes
\( n \) number of modes retained in the reconstruction
\( n_c \) number of velocity components
\( p_i(t) \) \( i^{th} \) unsteady pressure sensor
\( q \) total number of pressure sensors
\( Re \) Reynolds number based on chord length
\( t \) time stamp of PIV snapshot
\( T \) total number of PIV snapshots
\( U_\infty \) free stream velocity
\( u_i \) \( i^{th} \) velocity component
\( u, v, w \) streamwise - wall normal - spanwise velocity component
\( V \) spatial integration domain
\( x, y, z \) streamwise - wall normal - spanwise cartesian coordinate system
\( \alpha \) angle of attack

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I. Introduction

In flow control applications, specifically in closed-loop approaches, low-dimensional tools are needed to reduce the complexity of flows so that pertinent information can be extracted, estimated or treated in a real-time fashion. The Proper Orthogonal Decomposition is one such method and allows the user to decompose the flow field in a way that optimizes a quantity of fundamental interest for the control. In the largely used standard POD technique, as introduced by Lumley in 1967, the decomposition is performed on the velocity field and the kinetic energy is the optimized quantity. The first modes resulting from this standard decomposition display most of the energy contained in the flow.

Aubry et al. in 1988 were the first to use the POD to build a dynamical model for a turbulent flow. The model was derived through a Galerkin projection of the Navier-Stokes equations onto the POD modes and reproduced realistic dynamical behavior in the wall region of a turbulent boundary layer. Many numerical and experimental studies have followed which attempted to identify the dynamical characteristics of various flow configurations. The quality of each model depends on how well the POD modes relate to the dynamics. By choosing the right set of basis functions in the Galerkin projection, Noack & Eckelmann were able to analyze the cylinder wake flow transition. POD is also used in control applications, where it is typically part of an estimation technique to obtain a low-dimensional estimate of the flow state from experimental measurements. Glauser et al. have implemented a proportional closed-loop feedback control algorithm on the turbulent flow over a NACA-4412 airfoil using a combination of the standard POD technique and real-time surface pressure measurements. This combination is called the modified Linear Stochastic Measurement (mLSM). The authors were able to demonstrate the effectiveness of this closed-loop algorithm as it significantly delayed the flow separation over the airfoil and also displayed a large cost-reduction when compared to an open-loop approach.

However it is acknowledged that the standard POD technique can only result in a limited description of the flow dynamics since it is designed for energy optimization. Other modifications to the POD have been performed for flow configurations in which the kinetic energy might not be the main element of interest. Rowley et al. for instance developed a methodology to apply the POD technique to compressible flows by defining a new vector-valued variable that includes the thermodynamic variables as well as the velocity variables. In an attempt at capturing the vortex dynamics behind a backward-facing step flow, Kostas et al. performed the POD on both the vorticity field and the velocity field. They found that the vorticity decomposition not only captured the enstrophy (the now optimized quantity) more efficiently but also identified the coherent structures in the flow more clearly when compared to the vorticity field calculated from velocity-based POD modes. The standard POD has been designed to optimally capture the energy-containing structures in a flow field and although these largest structures do play an essential role in the flow, the truncated small-scales might still carry non-negligible dynamical information. Therefore, in any flow control application where an accurate knowledge of the dynamical evolution in time of the flow is of primary importance, a means of optimally representing the flow dynamics should be sought. Consequently, in this paper a new version of the POD is proposed - the convection POD or cPOD - which intends to better characterize the dynamical features of the flow field. A comparison between the velocity-based and the convection based POD formulations is presented and a discussion on the results and the expected improvements on the control method follows.

II. Experimental Setup

The experiment is conducted in the Syracuse University subsonic closed-loop wind tunnel on a NACA-4412 model airfoil. The wing along with the entire experimental setup described in this section is shown in Fig. 1. The airfoil has a 20 cm chord and a 61 cm span equal to the width of the test section to minimize end tip vortices. The flow speed is set at $U_\infty=10$ m/s, and the corresponding Reynolds number based on
chord length is Re=135,000. The experimental velocity measurements are acquired with a Dantec Dynamics stereo
scopic Particle Image Velocimetry system (PIV) which consists of two 1280×1024 pixel CCD cameras and a pair of pulsed New Wave Research 200 mJ Nd:YAG lasers. The design purposely allows the cameras and the wing to move together as the airfoil is pitched providing a fixed measurement window with respect to the airfoil. Generally, stereoscopic PIV systems are used to acquire all 3 components of velocity in a whole plane (3C-PIV). In the present experiment however, the cameras are positioned to take simultaneous velocity measurements in two adjoining planes, which results in a 2-component velocity field (2C-PIV) with a measurement window almost twice as large as in the case of a 3C-PIV configuration which allows for velocity measurements from upstream of the airfoil to the wake region. The measurement area is 31 cm in the streamwise direction by 13 cm in the wall-normal direction along with spatial resolutions of 2.0 mm and 2.4 mm which allows for accurate calculations of the spatial derivatives of velocity. A certain amount of noise is nonetheless introduced by the spatial derivative process, especially at the junction region between the two PIV windows. Denoising techniques are under study for extracting the numerical noise from the physical signal.

Figure 1. Overall view of the experimental setup and test section of the Syracuse University closed-loop subsonic wind tunnel.

The actuation used to control the flow consists of 14 piezoelectric diaphragms located near the leading edge of the airfoil. Each diaphragm is enclosed in an individual cavity, and the synthetic jet exit slots are evenly spaced along the span of the wing. Real-time flow measurements are available through 11 unsteady pressure sensors embedded along the chord at a mid-span position and evenly spaced between \( x/c = 0.29 \) and \( x/c = 0.78 \). Previous experimental investigations have shown that in the specific configuration where the laser plane is on top of the pressure array, the pressure measurements are biased by the sensors picking up the laser signature. Therefore the laser sheet is slightly shifted along the span away from the pressure array, though still covering a region where the 2D flow assumption pertains and strong correlations between the pressure and velocity signals still exist. The PIV trigger is recorded simultaneously with the 11 pressure signals and the PIV data to be able to compute cross-correlations between velocity fields and the pressure data. \( T=1000 \) statistically independent PIV velocity vector maps or snapshots were taken at \( \alpha = 10, 12, 14, 16, \) and \( 18^\circ \), control Off and control On. Each velocity map contains \( M=8757 \) vectors.

III. A new formulation of the Proper Orthogonal Decomposition

A. The standard POD

Two distinct but equivalent mathematical approaches can be considered when applying the POD technique to a database: the classical POD and the snapshot POD. The latter was introduced by Sirovich in 1987 and is essentially best suited for applications where the spatial resolution largely exceeds the temporal resolution. In the present case where velocity measurements are obtained using a PIV system, the high spatial resolution
justifies the use of the snapshot POD approach. The snapshot method reduces the problem to the dimension of \( T \) instead of \( n_c \times M \) in the classical POD method. Equations for both versions are laid out in Ausseur et al.,\(^\text{11}\) however in this paper only the snapshot version of the POD is presented.

In the original POD approach the data used to generate the orthonormal basis is the fluctuating velocity field \( \vec{u}(\vec{x},t) \). The latter is obtained by subtracting the ensemble average (performed on \( T \) snapshots) of the velocity field to the instantaneous realizations. The inner product of two vector variables \( \vec{f} \) and \( \vec{g} \) in \( L^2 \), the Hilbert space of square-integrable functions, is defined as:

\[
(\vec{f}, \vec{g}) = \int_V (f_1 g_1^* + f_2 g_2^* + f_3 g_3^*) d\vec{V}
\]

where \( V \) represents the spatial domain of integration. Applied to the fluctuating velocity variables, the norm is:

\[
||\vec{u}||^2 = (\vec{u}, \vec{u}) = \int_V (u^2 + v^2 + w^2) d\vec{V}
\]

Because the induced norm is twice the fluctuating kinetic energy of the flow as shown in the above expression, this makes this POD formulation very practical for many applications and accounts for its popularity in the turbulence field for the past years.

The following Fredholm integral eigenvalue problem is solved:

\[
\int_\tau C(t,t') a_n(t') dt' = \lambda_n a_n(t)
\]

where \( C(t,t') \) is the ensemble averaged temporal correlation matrix, defined as:

\[
C(t,t') = \frac{1}{T} \int_V u_i(\vec{x},t).u_i(\vec{x},t') d\vec{V}
\]

and \( a_n(t) \) are the temporal eigenfunctions. For reasons of consistency with the classical POD, the latter should be uncorrelated in time and their mean square value should equal their associated eigenvalue:

\[
< a^m.a^n >_\tau = \lambda^m \delta_{mn}
\]

The eigenvalue \( \lambda^n \) represents the individual contribution of its associated eigenvector \( \phi^n(\vec{x}) \) to the total fluctuating kinetic energy \( E \).

\[
E = \frac{1}{2} \sum_{n=1}^{N} \lambda^n
\]

The spatial eigenfunctions \( \phi^n(\vec{x}) \) must now be defined as:

\[
\phi^n_t(\vec{x}) = \frac{1}{T \lambda^n} \int_\tau a^n(t).u_i(\vec{x},t) dt
\]

so that they be orthonormal and verify:

\[
\int_V \phi^n(\vec{x}).\phi^n(\vec{x}) d\vec{V} = \delta_{mn}
\]

Finally, a reduced-order reconstruction of the velocity field is obtained using the expression:

\[
u_i(\vec{x},t) = \sum_{n=1}^{N_m} a^n(t) \phi^n_t(\vec{x})
\]

with \( N_m < N \) being the truncated number of modes and \( \sum_{n=1}^{N_m} \lambda^n / \sum_{j=1}^{N} \lambda^j \) the fraction of fluctuating kinetic energy retrieved in this low-dimensional flow representation. Figure 2 shows a reduced-order reconstruction, using 75 modes out of 1000, of the instantaneous velocity field in a fully separated flow case (\( \alpha = 16^\circ \)).
B. The convection POD

In most flow separation control applications, the control input is made in the boundary layer to force certain types of instabilities or to force specific shear layer shedding frequencies. When the control objective is reached, this has the effect of substantially modifying the flow dynamics (e.g. separation delay in the case of the flow over a wing, Karman vortex shedding reduction behind a cylinder, cavity tone suppression...). The important information for flow control therefore often lies downstream of where the control input is performed which is often in the separated shear layer (or boundary layer) region itself where the non-linear convecting structures determine the flow behavior. It must be emphasized that these structures are not always the most energetic part of the flow but their non-linear dynamics have a dramatic consequence on the flow state evolution. A greater knowledge of the instantaneous location of these structures and their strength could therefore be of great interest for flow control strategies.

Since the original POD eigenvectors are ‘only’ a basis optimized for the energy contained in the flow, it has been a controversy as to whether these modes really represent the structures present in the flow, and if they do, then how well can they describe their dynamics. Moreover, much interest has been geared towards linking fluid dynamics and modern controls through the development of low-order models of flows that are based on dynamical systems development through solving ordinary differential equations (ODE) derived from the Navier-Stokes equations. Similarly, dynamical systems can be obtained using the moments method where experimental data is used to "train" a set of ODEs, the form of which is based on the form of the Galerkin projection of the Navier-Stokes equations. These dynamical systems are very sensitive to initial conditions, the number of low dimensional modes used to reconstruct the system, and the quality of the training database. To improve the quality of the dataset and to bring knowledge of the non-linear convective information about the flow in the training of the dynamical system, one can imagine optimizing the orthogonal basis in terms of its dynamics instead of its energy content, in other words, construct a basis of the training database. To improve the quality of the dataset and to bring knowledge of the non-linear information, the number of low dimensional modes used to reconstruct the system, and the quality strength could therefore be of great interest for flow control strategies.

The induced norm from Eq. 11 (the square of the convection term) no longer is a known quantity as energy is to the POD or enstrophy to the vorticity based POD. However, the quantities \( u_i \frac{\partial u_i}{\partial x_i}, u_j \frac{\partial u_j}{\partial x_j}, u_k \frac{\partial u_k}{\partial x_k} \) are easily obtained experimentally by using PIV data which provides the high spatial resolution needed for the calculation of these terms.

\[
||u_i \frac{\partial \bar{u}}{\partial x_i}||^2 = (u_i \frac{\partial \bar{u}}{\partial x_i}, u_i \frac{\partial \bar{u}}{\partial x_i}) = \int_V [(u_i \frac{\partial u_i}{\partial x_i})^2 + (u_j \frac{\partial u_j}{\partial x_j})^2 + (u_k \frac{\partial u_k}{\partial x_k})^2]dV
\]

(11)

The convection POD kernel that is used in the integral eigenvalue problem from Eq. 3 is now as follows, with Einstein summation on i, j and k:

\[
C_{cPOD}(t, t') = \frac{1}{T} \int_V \sum_{i=1}^N u_{n,\frac{\partial u_i}{\partial x_i}}(\bar{x}, t)u_{n,\frac{\partial u_i}{\partial x_i}}(\bar{x}, t')dV
\]

(12)

From this kernel are derived the temporal expansion coefficient \( a^n_{cPOD}(t) \) and the spatial eigenfunctions \( \phi_{n,\frac{\partial u_i}{\partial x_i}}(\bar{x}) \) in the same fashion as in Eqs. 7 and 8 by substituting the velocity by the convection term.

The scalar fields for \( u_i \frac{\partial u_i}{\partial x_i} \) or \( u_i \frac{\partial u_i}{\partial x_i} \) can then be partially or totally rebuilt as follows:

\[
u_i \frac{\partial u_i}{\partial x_i}(\bar{x}, t) = \sum_{n=1}^N a^n_{cPOD}(t)\phi_{u_i \frac{\partial u_i}{\partial x_i}}(\bar{x})
\]

(13)

and

\[
u_i \frac{\partial v}{\partial x_i}(\bar{x}, t) = \sum_{n=1}^N a^n_{cPOD}(t)\phi_{u_k \frac{\partial u_k}{\partial x_k}}(\bar{x})
\]

(14)

Figure 3 shows a low dimensional reconstruction, using 75 modes out of 1000, of the convection term in a fully separated flow case.
IV. Results

A. Physical Interpretation

Both the velocity-based POD and the convection POD were performed using the two-component fluctuating velocity field above the wing. Figure 4 shows a comparison of the velocity-based modes and the convection based modes. The velocity-based POD eigenfunctions by definition show the average location of the energy-containing regions of the flow. As seen, the whole wake region of the wing exhibits high eigenfunction amplitudes and as the mode number increases, smaller and smaller energy containing scales are exhibited uniformly throughout the entire wake region. Although it is obvious that the velocity-based POD modes constitute an ideal basis for developing Galerkin models, the physical interpretation of these modes is however unconvincing to the eye. On the other hand, the convection POD eigenfunctions in this figure very clearly unveil convecting-like structures that grow as they evolve downstream both in the leading and trailing edge shear layers. Mode 1 alone (Fig. 4(b)) shows a whole series of structures in the upper and lower shear layers whereas standard POD mode 1 alone only pulls out the large recirculating region in the wake of the airfoil where most of the fluctuating energy lies. As the mode number increases in the cPOD, the apparent shedding frequency increases. Each instantaneous snapshot of the scalar convective field \( u_j \frac{\partial u_j}{\partial x_j} \) (or \( u_j \frac{\partial v_j}{\partial x_j} \)) is therefore a weighted combination of different wavenumber structures contained in each individual spatial mode (Eq. 13). This feature could be of great interest for flow control applications and is discussed further on. It is quite interesting to discover that even though the convection POD modes only contain time-averaged information about the structures in the shear layer (due to the Karhunen-Loève (KL) expansion procedure) a wave-like pattern appears in the eigenfunctions.

In this separated turbulent flow, both the unsteady and the convection terms in the material derivative of the velocity are important quantities balanced by the pressure gradient and the viscous forces (only close to the airfoil surface) as dictated by the Navier-Stokes equations (Eq. 10). The present PIV dataset does not permit a computation of the time derivative of the velocity but such a quantity could be of great interest for similar KL expansions where the kernel would now be of the form:

\[
C(t, t') = \frac{1}{T} \int_{V} \frac{\partial u_k}{\partial t}(\vec{x}, t) \frac{\partial u_k}{\partial t}(\vec{x}, t') d\vec{V}
\]  

In the limit of a steady flow where the material derivative was reduced to the non-linear convection term only, and assuming viscous effects are small away from the airfoil surface, then the convective term should be balanced by the pressure gradient alone. It is then possible to think of the structures visible in the convection POD modes as a succession of high and low pressure regions, as eddies convect by in the shear layer.

Figure 5 shows the evolution of the first spatial convection POD eigenfunction as the angle of attack \( \alpha \) is increased from an attached flow state (\( \alpha = 10^\circ \)) to fully separated (\( \alpha = 17.5^\circ \)). The flow separation process on the NACA-4412, as discussed in detail in Ausseur et al.\(^{11}\), starts at the trailing edge with a separation bubble that grows upstream until the leading edge separation occurs and massive reverse flow is experienced. It is interesting to notice the weakness of the convective structures when the boundary layer is fully attached (\( \alpha = 10^\circ \)). As the trailing edge separation bubble develops towards the leading edge (\( \alpha = 12 \) and \( 14^\circ \)) stronger convective structures are shed at the trailing edge. As the separated region becomes large (\( \alpha = 16 \) and \( 18^\circ \)), the leading edge vortex shedding becomes dramatically important and the trailing edge shedding frequency seems to decrease compared to lower angles of attack, which would be expected as the wing in this state acts like a bluff body with strong low frequency shedding. It is quite encouraging that the first convection POD mode alone is able to extract such physical events. When compared to the first standard POD mode that only extracts one average-like structure, convection mode 1 seems to extract several low-dimensional convecting-like structures from the flow.

B. Convergence and low-dimensional reconstructions

The usual way to quantify the efficiency of the KL expansion is by looking at the convergence of the cumulated eigenvalues and determining what percentage of the modes is necessary to retrieve the majority of the flow features. In laminar flows, the first few modes only will be able to capture close to 99% of the fluctuating energy in the flow. In the case of turbulent flows where the fluctuations around the mean are much larger,
it is common to need about 5% of the modes to capture 75% of the fluctuating energy, which still brings a significant reduction. The convergence of both the standard POD and the convection POD eigenvalues is shown in Fig. 6 in a cumulative form. These plots are however not strictly comparable since the eigenvalues in both cases represent different quantities. In the first case, the convergence of the fluctuating energy is plotted whereas in the second case, the convergence of the "convective energy" is shown. Looking at each convergence individually, the convection POD does not converge as fast as the standard POD, meaning that more modes are required to extract a given percentage of "convective energy". In this flow, only 50 standard POD modes (out of 1000) are enough to reach 75% of the fluctuating energy. In the case of the convection POD, around 350 modes are needed to reach such values, but it is yet undetermined how much of the "convective energy" is important to capture the main dynamics in the flow.

It is however possible to compare both POD methods on equal grounds by calculating the convection term from the velocity-based POD modes and then comparing them to the actual convection POD reconstruction. Figure 7 shows the original convection field, the convection POD reconstruction using 75 modes and the convection field calculated from the velocity-based POD also using 75 modes. It can be concluded from these figures that the flow information extracted by both methods is very different since the standard POD, though retrieving a significant amount of energy does not show any significant level of dynamics in the convective reconstruction. If it were possible to rebuild a low-dimensional velocity field from the convection POD reconstruction, it can be speculated that only a small amount of energy would be retrieved. This means that the most efficient modes in terms of energy are not the most efficient in terms of structure identification which can be partly explained by the fact (emphasized by Graham & Kevrekidis) that the Karhunen-Loève analysis through the ensemble averaging process ‘de-emphasizes intermittent events’ which are common in turbulent flows and can be dynamically very important. However, the convection POD through the calculation of the spatial derivatives of the velocity is able to exhibit what seem to be patterns of convective structures, as if a phase averaging had been performed.

Another feature pertaining to the convection POD is its ability to uncover, with a very low number of modes, a much more complex structure pattern. The aim of low-dimensional modeling is to represent simply (in terms of amount of information) a complex flow containing a certain number of coherent structures. Defining a structure is however subjective and the eye is probably a much more efficient tool than any other when it comes to recognizing coherent structures in a turbulent velocity field. It is also easier to decipher structures when moving in a reference frame with the flow. Therefore, in Figures 8(a) and (b), a velocity of $0.6 \times U_{\infty}$ has been subtracted from the instantaneous velocity field to reveal the structures in the shear layer. A 5-mode standard reconstruction of the velocity field shows a rather large vortex in the upper shear layer and its counterpart in the lower shear layer. In the original velocity field however, the upper shear layer seems to exhibit 3 large scale structures and 2 in the lower. As can be seen, these complex dynamics are extracted more efficiently in the convective reconstruction with the same number of modes. This shows promise for flow control applications where the least number of modes needed means the faster the control.

C. Application to Flow Control

As previously mentioned, the standard POD is very well suited for performing Galerkin projections of the Navier-Stokes equations and developing low-order dynamical systems for flows. In the case of a model of the flow using a basis constructed with the quantity $u_i \frac{\partial u}{\partial x}$, the field is wide open, and its potential seems promising. Rumpf performed a Galerkin projection of the vorticity equation onto a basis of eigenfunctions derived by taking the curl of the velocity eigenfunctions, thereby eliminating the pressure term. In the same way, a modified Galerkin projection of the NS equations onto a basis of convection modes could result in a model for the dynamics in the flow. However, a great amount of basic closed-loop control could be based on the cPOD since the control parameter, in our case, is the expansion coefficient $a_{n,\text{cPOD}}(t)$ that can be estimated instantaneously in the same manner using low-dimensional methods such as the modified Linear (or Quadratic) Stochastic Measurement as performed and detailed in Ausseur et al. This technique enables a direct measurement of the low-dimensional expansion coefficient $\tilde{a}_{n,\text{cPOD}}(t)$ from a small number of sensors located on the wing surface. For clarity purposes, the subscript denoting cPOD will be dropped from the following equations since they are valid in a general case. For each cPOD mode $n$, we can describe the estimated coefficient as a series expansion using the discrete instantaneous surface pressure measurements $p_i(t)$ at each streamwise position $i$ on the airfoil surface:
\[ \tilde{a}^n(t) = A^n_i p_i(t) + B^n_j p_j(t) + C^n_k p_k(t) + \ldots \quad i, j, k \in [1, q] \]  

This expansion can then be truncated above the linear term (mLSM) or the quadratic term (mQSM) to be able to estimate the expansion coefficient directly. To minimize the mean square error defined as,

\[ e_{a^n} = (\langle \tilde{a}^n(t) - a^n(t) \rangle)^2 \]

we solve the following matrix problem for \( A_{ni} \) in the linear estimation (mLSM):

\[
\begin{bmatrix}
\langle p_1p_1 \rangle & \cdots & \langle p_qp_1 \rangle \\
\vdots & \ddots & \vdots \\
\langle p_1p_q \rangle & \cdots & \langle p_qp_q \rangle \\
\end{bmatrix}
\begin{bmatrix}
A^n_1 \\
\vdots \\
A^n_q \\
\end{bmatrix} =
\begin{bmatrix}
\langle a^n p_1 \rangle \\
\vdots \\
\langle a^n p_q \rangle \\
\end{bmatrix}
\]

or solve the following matrix problem for \( A^n_i \) and \( B^n_{jk} \) in the quadratic estimation (mQSM):

\[
\begin{bmatrix}
\langle p_1p_1 \rangle & \cdots & \langle p_qp_1 \rangle & \langle p_1p_1p_1 \rangle & \cdots & \langle p_qp_qp_1 \rangle \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\langle p_1p_q \rangle & \cdots & \langle p_qp_q \rangle & \langle p_1p_1p_1 \rangle & \cdots & \langle p_qp_qp_q \rangle \\
\end{bmatrix}
\begin{bmatrix}
A^n_1 \\
\vdots \\
A^n_q \\
B^n_{11} \\
\vdots \\
B^n_{qq} \\
\end{bmatrix} =
\begin{bmatrix}
\langle a^n p_1 \rangle \\
\vdots \\
\langle a^n p_q \rangle \\
\langle a^n p_1p_1 \rangle \\
\vdots \\
\langle a^n p_qp_q \rangle \\
\end{bmatrix}
\]

Given \( A^n_i \), or \( A^n_i \) and \( B^n_{jk} \), the estimated expansion coefficients are then estimated in real-time using the following simple matrix multiplications:

\[
\text{mLSM : } \tilde{a}^n(t) = A^n_i p_i(t)
\]

\[
\text{mQSM : } \tilde{a}^n(t) = A^n_i p_i(t) + B^n_{jk} p_j(t)p_k(t)
\]

This estimation procedure is in the process of being performed on the NACA-4412 dataset.

D. Defining a new control objective

In the present case of flow separation control, one has to define a control objective based on the new convection POD coefficient. If the aim is to keep the boundary layer attached, the size of the shear layer structures must be kept at a minimum. Then, the objective will be to drive the lower modes to zero amplitude. On a case by case basis, one could also think of finding the mode that exhibits a specific shedding frequency of interest for flow control, like a sub-harmonic frequency in a cavity flow or a desired out-of-phase shedding frequency behind a bluff body that could reduce Karman vortex shedding. Then, the control objective becomes maximizing the specific mode of interest or driving all others to zero.

V. Conclusions and Perspectives

Both the standard POD and a new convection POD were performed on the NACA-4412 PIV database and results were qualitatively compared. It is shown that more dynamical information on the flow is extracted from the POD basis built from the convection term of the Navier-Stokes equations. Moreover, the information provided by the standard POD approach is of significantly different nature and does not seem to retain enough of the flow dynamics in the lower modes. A convincing shedding-like structure pattern is exhibited in all convection POD modes, with an apparent shedding frequency that increases with mode number. This characteristic can be of great interest to flow control applications where new control objectives could be envisioned.
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References

Figure 2. Instantaneous velocity field, $t=15$, $\alpha = 16^\circ$

Figure 3. Representation of the convection term $u_i \frac{\partial u_j}{\partial x_i}$, $t=15$, $\alpha = 16^\circ$
Figure 4. Comparison of the standard POD (left) versus cPOD (right) eigenvectors, modes 1, 5, 10 and 24 (top to bottom), $\alpha = 16^\circ$. 

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Figure 5. Comparison of the standard POD (left) and cPOD (right) eigenvectors, $n=1$, $\alpha = 10^\circ, 12^\circ, 14^\circ, 16^\circ$ and $18^\circ$ (top to bottom)
Figure 6. Convergence of the cumulated eigenvalues, $\alpha = 16^\circ$

(a) Standard POD

(b) cPOD

Figure 7. Comparison of convection field $u_i \frac{\partial u_i}{\partial x_i}$, $t=10$, $\alpha = 16^\circ$

(a) Convection POD, $n = 75$

(b) Standard POD, $n = 75$

(c) Original field
(a) Original velocity field in moving frame of reference

(b) Standard POD, n = 5, in moving frame of reference

(c) Original representation of $u_i \frac{\partial u_i}{\partial x_i}$

(d) Convection POD, n = 5

Figure 8. Comparison of a 5-mode low-dimensional reconstruction, t=52, $\alpha = 16^\circ$